ON THE EXTENDABILITY OF PROJECTIVE SURFACES AND A GENUS BOUND FOR ENRIQUES-FANO THREEFOLDS

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† *Dedicated to the memory of Giulia Cerutti, Olindo Ado Lopez and Saúl Sánchez*

Abstract. We introduce a new technique, based on Gaussian maps, to study the possibility, for a given surface, to lie on a threefold as a very ample divisor with given normal bundle. We give several applications, among which one to surfaces of general type and another one to Enriques surfaces. For the latter we prove that any threefold (with no assumption on its singularities) having as hyperplane section a smooth Enriques surface (by definition an Enriques-Fano threefold) has genus $g \leq 17$ (where g is the genus of its smooth curve sections). Moreover we find a new Enriques-Fano threefold of genus 9 whose normalization has canonical but not terminal singularities and does not admit $\mathbb{Q}$-smoothings.

1. Introduction

One of the most important contributions given in algebraic geometry in the last century, is the scheme of classification of higher dimensional varieties proposed by Mori theory. Despite the fact that many statements still remain conjectural, several beautiful theorems have been proved and the goal, at least in the birational realm, is particularly clear in dimension three: starting with a threefold X_0 with terminal singularities and using contractions of extremal rays, the Minimal Model Program (see e.g. [KM]) predicts to arrive either at a threefold X with K_X nef or at a Mori fiber space, that is ([R2]) there is an elementary contraction $X \to B$ with $\dim B < \dim X$. Arguably the simplest case of such spaces is when B is a point, that is X is a Fano threefold. As is well known, *smooth* Fano threefolds have been classified ([I1, I2, MM]), while, in the singular case, a classification, or at least a search for the numerical invariants, is still underway [Muk1, P1, JPR].

Both the old and recent works on the classification of smooth Fano threefolds use the important fact [Sh1] that a general anticanonical divisor is a smooth K3 surface. In [CLM1, CLM2] the authors studied varieties with canonical curve section and recovered, in a very simple way, using the point of view of Gaussian maps, a good part of the classification [Muk2]. The starting step of the latter method is Zak's theorem [Za, page 277] (see also [Lv, Thm.0.1]): If $Y \subset \mathbb{P}^r$ is a smooth variety of codimension at least two with normal bundle $N_{Y/\mathbb{P}^r}$ and $h^0(N_{Y/\mathbb{P}^r}(-1)) \leq r+1$, then the only variety $X \subset \mathbb{P}^{r+1}$ that has Y as hyperplane section is a cone over Y (when this happens $Y \subset \mathbb{P}^r$ is said to be *nonextendable*). Now the key point in the application of this theorem is to be able to calculate the cohomology of the normal bundle. This is of course an often difficult task, especially when the codimension of Y grows. It is here that Gaussian maps enter the picture by giving a big help in the case of curves [Wa, Prop.1.10]: if Y is a curve then

$$h^0(N_{Y/\mathbb{P}^r}(-1)) = r + 1 + \operatorname{cork} \Phi_{H_Y,\omega_Y} \tag{1}$$

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where Φ_{H_Y,ω_Y} is the Gaussian map associated to the canonical and hyperplane bundle H_Y of Y. For example when $X \subset \mathbb{P}^{r+1}$ is a smooth anticanonically embedded Fano threefold with general hyperplane section Y, in [CLM1, Thm.4 and Prop.3], $h^0(N_{Y/\mathbb{P}^r}(-1))$ was computed by calculating these coranks for the general curve section C of Y.

In the case above the proof was strongly based on the fact that C is a general curve on a general K3 surface and that the Hilbert scheme of K3 surfaces is essentially irreducible. On the other hand the latter fact is quite peculiar of K3 surfaces and we immediately realized that if one imposes different hyperplane sections to a threefold, for example Enriques surfaces, it becomes quite difficult to usefully rely on the curve section.

To study this and other cases it became apparent that it would be an important help to have an analogue of the formula (1) in higher dimension. We accomplish this in Section 2 by proving the following general result in the case of surfaces (a similar result holds in higher dimension):

Proposition 1.1. *Let $Y \subset \mathbb{P}^r$ be a smooth irreducible linearly normal surface and let H be its hyperplane bundle. Assume there is a base-point free and big line bundle D_0 on Y with $H^1(H-D_0) = 0$ and such that the general element $D \in |D_0|$ is not rational and satisfies*

(i) *the Gaussian map $\Phi_{H_D,\omega_D(D_0)}$ is surjective;*
(ii) *the multiplication maps μ_{V_D,ω_D} and $\mu_{V_D,\omega_D(D_0)}$ are surjective, where* $V_D := \mathrm{Im}\{H^0(Y, H - D_0) \to H^0(D, (H - D_0)_{|D})\}$.

Then

$$h^0(N_{Y/\mathbb{P}^r}(-1)) \le r + 1 + \operatorname{cork} \Phi_{H_D,\omega_D}.$$

Despite the apparent complexity of the above hypotheses, it should be mentioned that as soon as both D_0 and $H - D_0$ are base-point free and the degree of D is large with respect to its genus, the hypotheses are fulfilled unless D is hyperelliptic. Proposition 1.1 is therefore a flexible instrument to study threefolds whose hyperplane sections have large Picard group. This aspect complements in a nice way the recent work of Mukai [Muk1], where a classification of Gorenstein *indecomposable* Fano threefolds has been achieved: In fact indecomposable implies that a decomposition $H = D_0 + (H - D_0)$ with both D_0 and $H - D_0$ moving essentially does not exist.

As we will see in Section 3, Proposition 1.1 has several applications. A nice sample of this is the following consequence: a pluricanonical embedding of a surface of general type, and even some projection of it, is not, in many cases, hyperplane section of a threefold (different from a cone) (see also Remark 3.5 for sharpness). We recall that if Y is a minimal surface of general type containing no (-2)-curves, then mK_Y is very ample for $m \ge 5$ [Bo, Main Thm.].

Corollary 1.2. *Let $Y \subset \mathbb{P}V_m$ be a minimal surface of general type whose canonical bundle is base-point free and nonhyperelliptic and $V_m \subseteq H^0(mK_Y + \Delta)$ where $\Delta \ge 0$ and either Δ is nef or Δ is reduced and K_Y is ample. Suppose that either Y is regular or linearly normal and that*

$$m \ge \begin{cases} 9 & \text{if } K_Y^2 = 2; \\ 7 & \text{if } K_Y^2 = 3; \\ 6 & \text{if } K_Y^2 = 4 \text{ and the general curve in } |K_Y| \text{ is trigonal or if } K_Y^2 = 5 \text{ and} \\ & \text{the general curve in } |K_Y| \text{ is a plane quintic;} \\ 5 & \text{if either the general curve in } |K_Y| \text{ has Clifford index 2 or} \\ & 5 \le K_Y^2 \le 9 \text{ and the general curve in } |K_Y| \text{ is trigonal;} \\ 4 & \text{otherwise.} \end{cases}$$

Then Y is nonextendable.

Besides the mentioned applications, in the present article we will concentrate most of our attention on the case of Enriques-Fano threefolds: in analogy with Fano threefolds where an anticanonical divisor is a K3 surface, we define

Definition 1.3. *An* **Enriques-Fano threefold** *is an irreducible three-dimensional variety $X \subset \mathbb{P}^N$ having a hyperplane section S that is a smooth Enriques surface, and such that X is not a cone over S. We will say that X has genus g if g is the genus of its general curve section.*

Fano himself, in a 1938 article [Fa], claimed a classification of such threefolds, but his proof contains several gaps. Conte and Murre [CM] were the first to remark that an Enriques-Fano threefold must have some isolated singularities, typical examples of which are quadruple points with tangent cone the cone over the Veronese surface. Filling out some of the gaps in [Fa] and making some special assumptions on the singularities, Conte and Murre recovered some of the results of Fano, but not enough to give a classification, nor to bound the numerical invariants. On the opposite extreme, with the strong assumption that the Enriques-Fano threefold is a quotient of a *smooth* Fano threefold by an involution (this corresponds to having only cyclic quotient terminal singularities), a list was given by Bayle [Ba, Thm.A] and Sano [Sa, Thm.1.1], by using the classification of smooth Fano threefolds and studying which of them have such involutions.

Moreover, by the results of Minagawa [Mi, MainThm2], any Enriques-Fano threefold with at most terminal singularities admits a $\mathbb{Q}$-smoothing, that is [Mi, R1], it appears as central fiber of a small deformation over the 1-parameter unit disk, such that a general fiber has only cyclic quotient terminal singularities. This, together with the results of Bayle and Sano, gives then the bound $g \leq 13$ for Enriques-Fano threefolds with at most terminal singularities.

Bayle and Sano recovered all of the examples of Enriques-Fano threefolds given by Fano and Conte-Murre. As these were the only known examples, it has been conjectured for some time now that this list is complete or, at least, that the genus is bounded, in analogy with the celebrated genus bound for smooth Fano threefolds [I1, I2, Sh2].

In Section 16, we will show that the list of known Enriques-Fano threefolds of Fano, Conte-Murre, Bayle and Sano *is not complete* (in fact not even after specialization), by finding a new Enriques-Fano threefold enjoying several peculiar properties (for a more precise statement and related questions, see Proposition 16.1 and Remark 16.2):

Proposition 1.4. *There exists an Enriques-Fano threefold $X \subset \mathbb{P}^9$ of genus 9 such that neither X nor its polarized normalization belongs to the list of Fano-Conte-Murre-Bayle-Sano.*

Moreover, X does not have a $\mathbb{Q}$-smoothing and in particular X is not in the closure of the component of the Hilbert scheme made of Fano-Conte-Murre-Bayle-Sano's examples. Its normalization $\widetilde{X}$ has canonical but not terminal singularities and does not admit $\mathbb{Q}$-smoothings.

Observe that $\widetilde{X}$ is a $\mathbb{Q}$-Fano threefold of Fano index 1 with canonical singularities not having a $\mathbb{Q}$-smoothing, thus showing that Minagawa's theorem [Mi, MainThm.2] cannot be extended to the canonical case.

In the present article we apply Proposition 1.1 to get a genus bound on Enriques-Fano threefolds, with no assumption on their singularities:

Theorem 1.5. *Let $X \subset \mathbb{P}^r$ be an Enriques-Fano threefold of genus g. Then $g \leq 17$.*

A more precise result for $g = 15$ and 17 is proved in Proposition 15.1.

We remark that very recently Prokhorov [P1, P2] proved the same genus bound $g \leq 17$ for Enriques-Fano threefolds and at the same time constructed an example of a new Enriques-Fano threefold of genus 17 [P2, Prop.3.2], thus showing that the bound $g \leq 17$ is in fact optimal. His methods are completely different from ours, in that he uses the log minimal model program in the category of G-varieties and results about singularities of Enriques-Fano threefolds of Cheltsov [Ch].

On the other hand, our procedure relies only on the geometry of curves on Enriques surfaces (see also Remark 16.3). In any case, both our example in Proposition 1.4 and Prokhorov's new examples (in fact, he also gives a new example in genus 13), shows that new methods were required in the classification of Enriques-Fano threefolds with arbitrary singularities.

Now a few words on our method of proof. In Section 4 we review some basic results that will be needed in our study of Enriques surfaces. In Section 5 we apply Proposition 1.1 to Enriques surfaces and obtain the main results on nonextendability needed in the rest of the article (Propositions 5.1, 5.2, 5.4 and 5.5). In Section 6 we prove Theorem 1.5 for all Enriques-Fano threefolds except for some concrete embedding line bundles on the Enriques surface section. These are divided into different groups and then handled one by one in Sections 7-14 by finding suitable divisors satisfying the conditions of Proposition 1.1, thus allowing us to prove our main theorem and a more precise statement for $g = 15$ and 17 in Section 15.

To prove our results it turns out that one needs effective criteria to ensure the surjectivity of Gaussian maps on curves on Enriques surfaces and of multiplication maps of (not always complete) linear systems on such curves. To handle the first problem a good knowledge of the Brill-Noether theory of a curve lying on an Enriques surface and general in its linear system must be available. We studied this independent problem in another article ([KL2]) and consequently obtained results ensuring the surjectivity of Gaussian maps in [KL3] (see also Theorem 5.3). To handle the multiplication maps, we find an effective criterion in Lemma 5.7 (which holds on any surface) in the present article.

2. Normal bundle estimates

We devise in this section a general method to give an upper bound on the cohomology of the normal bundle of an embedded variety. We state it here only in the case of surfaces to avoid a lengthy list of conditions.

Notation 2.1. Let L, M be two line bundles on a smooth projective variety. Given $V \subseteq H^0(L)$ we will denote by $\mu_{V,M} : V \otimes H^0(M) \longrightarrow H^0(L \otimes M)$ the multiplication map of sections, $\mu_{L,M}$ when $V = H^0(L)$, by $R(L, M)$ the kernel of $\mu_{L,M}$ and by $\Phi_{L,M} : R(L, M) \longrightarrow H^0(\Omega^1_X \otimes L \otimes M)$ the Gaussian map (that can be defined locally by $\Phi_{L,M}(s \otimes t) = sdt - tds$, see [Wa, 1.1]).

Proof of Proposition 1.1. To estimate $h^0(N_{Y/\mathbb{P}^r}(-1))$ we will use the exact sequence

$$0 \longrightarrow N_{Y/\mathbb{P}^r}(-D_0 - H) \longrightarrow N_{Y/\mathbb{P}^r}(-H) \longrightarrow N_{Y/\mathbb{P}^r}(-H)_{|D} \longrightarrow 0$$

and prove that

$$h^0(N_{Y/\mathbb{P}^r}(-D_0 - H)) = 0 \tag{2}$$

and

$$h^0(N_{Y/\mathbb{P}^r}(-H)_{|D}) \leq r + 1 + \operatorname{cork} \Phi_{H_D,\omega_D}. \tag{3}$$

To prove (2), let us see first that it is enough to have

$$h^0(N_{Y/\mathbb{P}^r}(-D_0 - H)_{|D}) = 0 \ \text{ for a general } D \in |D_0|. \tag{4}$$

In fact, by hypothesis there is a nonempty open subset $U \subseteq |D_0|$ such that every $D \in U$ is smooth irreducible and satisfies (i) and (ii). Now if $H^0(N_{Y/\mathbb{P}^r}(-D_0 - H))$ has a nonzero section σ, then, as $h^0(D_0) \geq 2$, there is a nonempty open subset $U_\sigma \subseteq U$ such that, for every $D \in U_\sigma$, we can find a point $x \in D$ with $\sigma(x) \neq 0$. The latter, of course, contradicts (4).

Now (4) follows from the exact sequence

$$0 \longrightarrow N_{D/Y}(-D_0 - H) \longrightarrow N_{D/\mathbb{P}^r}(-D_0 - H) \longrightarrow N_{Y/\mathbb{P}^r}(-D_0 - H)_{|D} \longrightarrow 0$$

and the two conditions

$$h^0(N_{D/\mathbb{P}^r}(-D_0-H))=0, \tag{5}$$

$$\varphi_{H+D}: H^1(N_{D/Y}(-D_0-H)) \longrightarrow H^1(N_{D/\mathbb{P}^r}(-D_0-H)) \text{ is injective.} \tag{6}$$

To see (5), we note that the multiplication map $\mu_{H_D,\omega_D(D_0)}$ is surjective by the H^0-lemma [Gr, Thm.4.e.1], since $|D_{0|D}|$ is base-point free, whence $D_0^2 \geq 2$, therefore $h^1(\omega_D(D_0-H)) = h^0((H-D_0)_{|D}) \leq h^0(H_D)-2$, as H_D is very ample. Now let $\mathbb{P}^k \subseteq \mathbb{P}^r$ be the linear span of D. The exact sequence

$$0 \longrightarrow N_{D/\mathbb{P}^k}(-D_0-H) \longrightarrow N_{D/\mathbb{P}^r}(-D_0-H) \longrightarrow \mathcal{O}_D(-D_0)^{\oplus(r-k)} \longrightarrow 0 \tag{7}$$

and the hypothesis $D_0^2>0$ imply that $h^0(N_{D/\mathbb{P}^r}(-D_0-H)) = h^0(N_{D/\mathbb{P}^k}(-D_0-H))$. Since Y is linearly normal and $H^1(H-D_0)=0$, we have that also D is linearly normal. As $\mu_{H_D,\omega_D(D_0)}$ is surjective, by [Wa, Prop.1.10], we get that $h^0(N_{D/\mathbb{P}^k}(-D_0-H)) = \operatorname{cork}\Phi_{H_D,\omega_D(D_0)} = 0$ because of (i), and this proves (5). As for (6), we prove the surjectivity of φ^*_{H+D} with the help of the commutative diagram

$$\begin{tikzcd} H^0(\mathcal{I}_{D/\mathbb{P}^r}(H)) \otimes H^0(\omega_D(D_0)) \arrow[r] \arrow[d, "f"] & H^0(N^*_{D/\mathbb{P}^r} \otimes \omega_D(D_0+H)) \arrow[d, "\varphi^*_{H+D}"] \\ H^0(\mathcal{I}_{D/Y}(H)) \otimes H^0(\omega_D(D_0)) \arrow[r, "h"] & H^0(N^*_{D/Y} \otimes \omega_D(D_0+H)). \end{tikzcd} \tag{8}$$

Here f is surjective by the linear normality of Y, while h factorizes as

$$H^0(\mathcal{J}_{D/Y}(H)) \otimes H^0(\omega_D(D_0)) \twoheadrightarrow V_D \otimes H^0(\omega_D(D_0)) \stackrel{\mu_{V_D,\omega_D(D_0)}}{\longrightarrow} H^0(N^*_{D/Y} \otimes \omega_D(D_0+H)),$$

whence also h is surjective by (ii).

Finally, to prove (3), recall that the multiplication map μ_{H_D,ω_D} is surjective by [AS, Thm.1.6] since D is not rational, whence $h^0(N_{D/\mathbb{P}^k}(-H)) = k+1+\operatorname{cork}\Phi_{H_D,\omega_D}$ by [Wa, Prop.1.10]. Therefore, twisting (7) by $\mathcal{O}_D(D_0)$, we get $h^0(N_{D/\mathbb{P}^r}(-H)) \leq r+1+\operatorname{cork}\Phi_{H_D,\omega_D}$ and (3) will now follow by the exact sequence

$$0 \longrightarrow N_{D/Y}(-H) \longrightarrow N_{D/\mathbb{P}^r}(-H) \longrightarrow N_{Y/\mathbb{P}^r}(-H)_{|D} \longrightarrow 0$$

and the injectivity of $\varphi_H : H^1(N_{D/Y}(-H)) \longrightarrow H^1(N_{D/\mathbb{P}^r}(-H))$. The latter follows, as in (8), from the commutative diagram

$$\begin{tikzcd} H^0(\mathcal{I}_{D/\mathbb{P}^r}(H)) \otimes H^0(\omega_D) \arrow[r] \arrow[d] & H^0(N^*_{D/\mathbb{P}^r} \otimes \omega_D(H)) \arrow[d, "\varphi^*_H"] \\ H^0(\mathcal{I}_{D/Y}(H)) \otimes H^0(\omega_D) \arrow[r] & H^0(N^*_{D/Y} \otimes \omega_D(H)) \end{tikzcd}$$

by the linear normality of Y and the surjectivity of μ_{V_D,ω_D}. □

Remark 2.2. In the above proposition and also in Corollary 2.4 below, the surjectivity of $\mu_{V_D,\omega_D(D_0)}$ can be replaced by either one of the following

(i) the multiplication map $\mu_{\omega_D(H-D_0),D_{0|D}}$ is surjective;
(ii) $h^0((2D_0-H)_{|D}) \leq h^0(D_{0|D})-2$;
(iii) $H.D_0 > 2D_0^2$.

Proof. The commutative diagram

$$\begin{array}{ccc} V_D \otimes H^0(\omega_D) \otimes H^0(D_{0|D}) & \xrightarrow{\mu_{V_D,\omega_D}\otimes \mathrm{Id}} & H^0(\omega_D(H-D_0)) \otimes H^0(D_{0|D}) \\ \downarrow & & \downarrow \mu_{\omega_D(H-D_0),D_{0|D}} \\ V_D \otimes H^0(\omega_D(D_0)) & \xrightarrow{\mu_{V_D,\omega_D(D_0)}} & H^0(\omega_D(H)) \end{array}$$

and the assumed surjectivity of μ_{V_D,ω_D}, show that (i) is enough. Now (ii) implies (i), by the H^0-lemma [Gr, Thm.4.e.1], while, under hypothesis (iii), we have that $h^0((2D_0-H)_{|D}) = 0$, whence (ii) holds.

Remark 2.3. The important conditions, in Proposition 1.1, are the surjectivity of μ_{V_D,ω_D} and the control of the corank of Φ_{H_D,ω_D}. On the Gaussian map side it must be said that all the known results imply surjectivity under some conditions (more or less of the type $H.D >> g(D)$), but no good bound on the corank is in general known. On the other hand, in many applications, one of the most important advantages is that one can reduce to numerical conditions involving H and D_0 (see for example Proposition 5.2).

The upper bound provided by Proposition 1.1 can be applied in many instances to control how many times Y can be extended to higher dimensional varieties. However the most interesting and useful application will be to one simple extension.

Corollary 2.4. *Let $Y \subset \mathbb{P}^r$ be a smooth irreducible surface which is either linearly normal or regular (that is, $h^1(\mathcal{O}_Y) = 0$) and let H be its hyperplane bundle. Assume there is a base-point free and big line bundle D_0 on Y with $H^1(H - D_0) = 0$ and such that the general element $D \in |D_0|$ is not rational and satisfies*

(i) *the Gaussian map Φ_{H_D,ω_D} is surjective;*
(ii) *the multiplication maps μ_{V_D,ω_D} and $\mu_{V_D,\omega_D(D_0)}$ are surjective, where*
$V_D := \mathrm{Im}\{H^0(Y, H - D_0) \to H^0(D, (H - D_0)_{|D})\}$.

Then Y is nonextendable.

Proof. Note that $g(D) \geq 2$, else Φ_{H_D,ω_D} is not surjective. Also since $\mu_{V_D,\omega_D(D_0)}$ is surjective, we must have that V_D (whence also $|(H - D_0)_{|D}|$) is base-point free, as $|\omega_D(H)|$ is such. Therefore $2g(D)-2+(H-D_0).D > 0$, whence $h^1(\omega_D^2(H-D_0)) = 0$ and the H^0-lemma [Gr, Thm.4.e.1] implies that the multiplication map $\mu_{\omega_D^2(H),D_{0|D}}$ is surjective. Now (i) and the commutative diagram

$$\begin{array}{ccc} R(H_D,\omega_D) \otimes H^0(D_{0|D}) & \xrightarrow{\Phi_{H_D,\omega_D}\otimes \mathrm{Id}} & H^0(\omega_D^2(H)) \otimes H^0(D_{0|D}) \\ \downarrow & & \downarrow \mu_{\omega_D^2(H),D_{0|D}} \\ R(H_D,\omega_D(D_0)) & \xrightarrow{\Phi_{H_D,\omega_D(D_0)}} & H^0(\omega_D^2(H+D_0)) \end{array}$$

give that also $\Phi_{H_D,\omega_D(D_0)}$ is surjective.

If Y is linearly normal the result therefore follows by Zak's theorem [Za, page 277], [Lv, Thm.0.1], and Proposition 1.1.

Assume now that $h^1(\mathcal{O}_Y) = 0$ and that $Y \subset \mathbb{P}^r$ is extendable, that is, there exists a nondegenerate threefold $X \subset \mathbb{P}^{r+1}$ which is not a cone over Y and such that $Y = X \cap \mathbb{P}^r$ is a hyperplane section. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities and let $L = \pi^*\mathcal{O}_X(1)$ and $\widetilde{Y} = \pi^{-1}(Y)$. Since Y is smooth we have $Y \cap \operatorname{Sing} X = \emptyset$, whence there is an isomorphism $(\widetilde{Y}, L_{|\widetilde{Y}}) \cong (Y, \mathcal{O}_Y(1))$. Now L is

nef and birational, whence $H^1(\widetilde{X}, -L) = 0$ by Kawamata-Viehweg vanishing. Moreover, as $\widetilde{Y} \in |L|$, we have, for all k, an exact sequence

$$0 \to kL \to (k+1)L \to (k+1)L_{|\widetilde{Y}} \to 0.$$

Setting $k = -1$ we get that $H^1(\mathcal{O}_{\widetilde{X}}) \subseteq H^1(\mathcal{O}_{\widetilde{Y}}) = H^1(\mathcal{O}_Y) = 0$, therefore, setting $k = 0$, we deduce the surjectivity of the restriction map $H^0(\widetilde{X}, L) \to H^0(\widetilde{Y}, L_{|\widetilde{Y}})$.

Consider the birational map $\varphi_L : \widetilde{X} \to \mathbb{P}^N$ where $N \geq r+1$, let $\overline{X} = \varphi_L(\widetilde{X})$ and let $\overline{Y}$ be the hyperplane section of $\overline{X}$ corresponding to $\widetilde{Y} \in |L|$. Now $\overline{Y} \cong Y$ and the embedding $\overline{Y} \subset \mathbb{P}^{N-1}$ is given by the complete linear series $|\mathcal{O}_Y(1)|$. Note also that, by construction, $\overline{X} \subset \mathbb{P}^N$ projects to $X \subset \mathbb{P}^{r+1}$, whence $\overline{X}$ is not a cone over $\overline{Y}$. Therefore $\overline{Y} \subset \mathbb{P}^{N-1}$ is linearly normal and extendable. But also on $\overline{Y}$ we have a line bundle $\overline{D_0}$ satisfying the same properties as D_0, whence, by the proof in the linearly normal case, $\overline{Y}$ is nonextendable, a contradiction. □

3. Absence of Veronese embeddings on threefolds

It was already known to Scorza in 1909 [Sc] that the Veronese varieties $v_m(\mathbb{P}^n)$ are nonextendable for $m > 1$ and $n > 1$. For a Veronese embedding of any variety we can use Zak's theorem to deduce nonextendability, as follows (we omit the case of curves that can be done, as is well-known, via Gaussian maps)

Remark 3.1. *Let $X \subset \mathbb{P}^r$ be a smooth irreducible nondegenerate n-dimensional variety, $n \geq 2$, $L = \mathcal{O}_X(1)$ and let $\varphi_{mL}(X) \subset \mathbb{P}^N$ be the m-th Veronese embedding of X.*

If $H^1(T_X(-mL)) = 0$ then $\varphi_{mL}(X)$ is nonextendable. In particular the latter holds if

$$m > \max\{2, n+2+\frac{K_X.L^{n-1} - 2r + 2n + 2}{L^n}\}.$$

Proof. Set $Y = \varphi_{mL}(X)$. From the exact sequences

$$0 \longrightarrow \mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y^{\oplus(N+1)} \longrightarrow T_{\mathbb{P}^N}(-1)_{|Y} \longrightarrow 0$$

$$0 \longrightarrow T_Y(-1) \longrightarrow T_{\mathbb{P}^N}(-1)_{|Y} \longrightarrow N_{Y/\mathbb{P}^N}(-1) \longrightarrow 0$$

and Kodaira vanishing we deduce that $h^0(N_{Y/\mathbb{P}^N}(-1)) \leq h^0(T_{\mathbb{P}^N}(-1)_{|Y}) + h^1(T_Y(-1)) = N + 1 + h^1(T_X(-mL)) = N+1$, and it just remains to apply Zak's theorem [Za, page 277], [Lv, Thm.0.1].

To see the last assertion observe that since $n \geq 2$ and $m \geq 3$ we have, as is well-known, $h^1(T_X(-mL)) = h^0(N_{X/\mathbb{P}^r}(-mL))$. Now if the latter were not zero, the same would hold for a general hyperplane section $X \cap H$ of X and so on until the curve section $C \subset \mathbb{P}^{r-n+1}$. Now taking $r-n-1$ general points $P_j \in C$, we have an exact sequence [BEL, 2.7]

$$0 \longrightarrow \bigoplus_{j=1}^{r-n-1} \mathcal{O}_C(1-m)(2P_j) \longrightarrow N_{C/\mathbb{P}^{r-n+1}}(-m) \longrightarrow \omega_C(3-m)(-2\sum_{j=1}^{r-n-1} P_j) \longrightarrow 0$$

whence $h^0(N_{C/\mathbb{P}^{r-n+1}}(-m)) = 0$ for reasons of degree. □

In the case of surfaces, as an application of Corollary 2.4, we can give an extension of the above remark to multiples of big and nef line bundles.

Definition 3.2. *Let Y be a smooth surface and let L be an effective line bundle on Y such that the general divisor $D \in |L|$ is smooth and irreducible. We say that L is* **hyperelliptic, trigonal**, *etc.,*

if D is such. We denote by $\operatorname{Cliff}(L)$ *the Clifford index of D. Moreover, when $L^2 > 0$, we set*

$$\varepsilon(L) = \begin{cases} 3 & \text{if } L \text{ is trigonal;} \\ 5 & \text{if } \operatorname{Cliff}(L) \geq 3; \\ 0 & \text{if } \operatorname{Cliff}(L) = 2. \end{cases}$$

and

$$m(L) = \begin{cases} \frac{16}{L^2} & \text{if } L.(L+K_Y) = 4; \\ \frac{25}{L^2} & \text{if } L.(L+K_Y) = 10 \text{ and the general divisor in } |L| \text{ is a plane quintic;} \\ \frac{3L.K_Y+18}{2L^2} + \frac{3}{2} & \text{if } 6 \leq L.(L+K_Y) \leq 22 \text{ and } L \text{ is trigonal;} \\ \frac{2L.K_Y - \varepsilon(L)}{L^2} + 2 & \text{otherwise.} \end{cases}$$

Corollary 3.3. *Let $Y \subset \mathbb{P}V$ be a smooth surface with $V \subseteq H^0(mL+\Delta)$, where L is a base-point free, big, nonhyperelliptic line bundle on Y with $L.(L+K_Y) \geq 4$ and $\Delta \geq 0$ is a divisor. Suppose that either Y is regular or linearly normal and that m is such that $H^1((m-2)L+\Delta) = 0$ and $m > \max\{m(L) - \frac{L.\Delta}{L^2}, \lceil \frac{L.K_Y+2-L.\Delta}{L^2} \rceil + 1\}$. Then Y is nonextendable.*

Proof. We apply Corollary 2.4 with $D_0 = L$ and $H = mL + \Delta$. By hypothesis the general $D \in |L|$ is smooth and irreducible of genus $g(D) = \frac{1}{2}L.(L+K_Y)+1$. Since $H^1(H-2L) = 0$, we have $V_D = H^0((H-L)_{|D})$. Also $(H-L).D = (m-1)L^2 + L.\Delta \geq L.(L+K_Y)+2 = 2g(D)$ by hypothesis, whence $|(H-L)_{|D}|$ is base-point free and birational (as D is not hyperelliptic) and the multiplication map μ_{V_D,ω_D} is surjective by [AS, Thm.1.6]. Moreover $H^1((H-L)_{|D}) = 0$, whence also $H^1(H-L) = 0$ by the exact sequence

$$0 \longrightarrow H - 2L \longrightarrow H - L \longrightarrow (H-L)_{|D} \longrightarrow 0.$$

The surjectivity of $\mu_{V_D,\omega_D(L)}$ now follows by [Gr, Cor.4.e.4] since $\deg \omega_D(L) \geq 2g(D)+1$ because $L^2 \geq 3$: If $L^2 \leq 2$ we have that $h^0(L_{|D}) \leq 1$ as D is not hyperelliptic, whence $h^0(L) \leq 2$, contradicting the hypotheses on L. Finally the surjectivity of Φ_{H_D,ω_D} follows by the inequality $m > m(L) - \frac{L.\Delta}{L^2}$ and well-known results about Gaussian maps ([Wa, Prop.1.10], [KL3, Prop.2.9, Prop.2.11 and Cor.2.10], [BEL, Thm.2]). □

Remark 3.4. The above result does apply, in some instances, already for $m = 1$ or 2. Also observe that the base-point free ample and hyperelliptic line bundles are essentially classified by several results in adjunction theory (see [BS] and references therein).

We can be a little bit more precise in the interesting case of pluricanonical embeddings.

Proof of Corollary 1.2. We apply Corollary 3.3 with $L = K_Y$ and $H = mK_Y + \Delta$, and we just need to check that $H^1((m-2)K_Y + \Delta) = 0$. If Δ is nef this follows by Kawamata-Viehweg vanishing. Now suppose that Δ is reduced and K_Y is ample. Again by Kawamata-Viehweg vanishing we have that $H^1((m-2)K_Y) = 0$, whence the exact sequence

$$0 \longrightarrow (m-2)K_Y \longrightarrow (m-2)K_Y + \Delta \longrightarrow \mathcal{O}_\Delta((m-2)K_Y + \Delta) \longrightarrow 0$$

shows that $H^1((m-2)K_Y+\Delta) = 0$ since $h^1(\mathcal{O}_\Delta((m-2)K_Y+\Delta)) = h^0(\mathcal{O}_\Delta(-(m-3)K_Y)) = 0$. □

Remark 3.5. Consider the 5-uple embedding X of $\mathbb{P}^3$ into $\mathbb{P}^{55}$. A general hyperplane section of X is a smooth surface Y embedded with $5K_Y$ and satisfying $K_Y^2 = 5$. Also consider the 4-uple embedding of a smooth quadric hypersurface in $\mathbb{P}^4$ into $\mathbb{P}^{54}$. Its general hyperplane section is a smooth surface Y embedded with $4K_Y$ and satisfying $K_Y^2 = 8$. Therefore, in general, the conditions on K_Y^2 and m cannot be weakened.

Remark 3.6. If K_Y is hyperelliptic, then $2K_Y$ is not birational and these surfaces have been classified by the work of several authors (see [BCP] and references therein).

We can be even more precise in the interesting case of adjoint embeddings.

Corollary 3.7. *Let $Y \subset \mathbb{P}V$ be a minimal surface of general type with base-point free and nonhyperelliptic canonical bundle and $V \subseteq H^0(K_Y + L + \Delta)$, where L is a line bundle on Y and $\Delta \geq 0$ is a divisor. Suppose that Y is either regular or linearly normal, that $H^1(L + \Delta - K_Y) = 0$ and that*

$$L.K_Y + K_Y.\Delta > \begin{cases} 14 & \text{if } K_Y^2 = 2; \\ 20 & \text{if } K_Y^2 = 5 \text{ and the general divisor in } |K_Y| \text{ is a plane quintic;} \\ 2K_Y^2 + 9 & \text{if } 3 \leq K_Y^2 \leq 11 \text{ and } K_Y \text{ is trigonal;} \\ 3K_Y^2 - \varepsilon(K_Y) & \text{otherwise.} \end{cases}.$$

Then Y is nonextendable.

Proof. Similar to the proof of Corollary 3.3 with $D_0 = K_Y$ and $H = K_Y + L + \Delta$. □

To state the pluriadjoint case, given a big line bundle L on a smooth surface Y we define the function

$$\nu(L) = \begin{cases} \frac{12}{L^2} + 1 & \text{if } L.(L + K_Y) = 4; \\ \frac{15}{L^2} + 1 & \text{if } L.(L + K_Y) = 10 \text{ and the general divisor in } |L| \text{ is a plane quintic;} \\ \frac{L.K_Y + 18}{2L^2} + \frac{3}{2} & \text{if } 6 \leq L.(L + K_Y) \leq 22 \text{ and } L \text{ is trigonal;} \\ \frac{L.K_Y - \varepsilon(L)}{L^2} + 2 & \text{otherwise.} \end{cases}$$

Corollary 3.8. *Let $Y \subset \mathbb{P}V$ be a smooth surface with $V \subseteq H^0(K_Y + mL + \Delta)$ where L is a base-point free, big and nonhyperelliptic line bundle on Y with $L.(L + K_Y) \geq 4$ and $\Delta \geq 0$ is a divisor such that $H^1(K_Y + (m-2)L + \Delta) = 0$. Suppose that Y is either regular or linearly normal and that $m > \max\{2 + \frac{1}{L^2}, \nu(L)\} - \frac{L.\Delta}{L^2}$. Then Y is nonextendable.*

Proof. Similar to the proof of Corollary 3.3 with $D_0 = L$ and $H = K_Y + mL + \Delta$. □

4. Basic results on line bundles on Enriques surfaces

Definition 4.1. *Let S be an Enriques surface. We denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors (or line bundles) on S. A line bundle L is* **primitive** *if $L \equiv hL'$ for some line bundle L' and some integer h, implies $h = \pm 1$. A* **nodal** *curve on S is a smooth rational curve. A* **nodal cycle** *is a divisor $R > 0$ such that, for any $0 < R' \leq R$ we have $(R')^2 \leq -2$. An* **isotropic divisor** *F on S is a divisor such that $F^2 = 0$ and $F \not\equiv 0$. An* **isotropic k-sequence** *is a set $\{f_1, \dots, f_k\}$ of isotropic divisors such that $f_i.f_j = 1$ for $i \neq j$.*

We will often use the fact that if R is a nodal cycle, then $h^0(\mathcal{O}_S(R)) = 1$ and $h^0(\mathcal{O}_S(R + K_S)) = 0$.

Let L be a line bundle on S with $L^2 > 0$. Following [CD] we define

$$\phi(L) = \inf\{|F.L| \ : \ F \in \operatorname{Pic} S, F^2 = 0, F \not\equiv 0\}.$$

Two important properties of this function, which will be used throughout the article, are that $\phi(L)^2 \leq L^2$ [CD, Cor.2.7.1] and that, if L is nef, then there exists a genus one pencil $|2E|$ such that $E.L = \phi(L)$ ([Co, 2.11] or by [CD, Cor.2.7.1, Prop.2.7.1 and Thm.3.2.1]). Moreover we will extensively use (often without further mentioning) the fact that a nef line bundle L with $L^2 \geq 4$ is base-point free if and only if $\phi(L) \geq 2$ [CD, Prop.3.1.6, 3.1.4 and Thm.4.4.1].

Lemma 4.2. *Let S be an Enriques surface, let L be a line bundle on S such that $L > 0$ and $L^2 > 0$ and let F be an effective divisor on S such that $F^2 = 0$ and $\phi(L) = |F.L|$. Moreover let A, B be two effective divisors on S such that $A^2 \geq 0$ and $B^2 \geq 0$. Then*

(a) *$F.L > 0$;*
(b) *if α is a positive integer such that $(L - \alpha F)^2 \geq 0$, then $L - \alpha F > 0$;*
(c) *$A.B \geq 0$ with equality if and only if there exists a primitive divisor $D > 0$ and integers $a \geq 1, b \geq 1$ such that $D^2 = 0$ and $A \equiv aD, B \equiv bD$.*

Proof. For parts (a) and (b) see [KL2, Lemma2.5]. Part (c) is proved in [KL1, Lemma2.1]. □

We will often use the ensuing

Lemma 4.3. *For $1 \leq i \leq 4$ let $F_i > 0$ be four isotropic divisors such that $F_1.F_2 = F_3.F_4 = 1$ and $F_1.F_3 = F_2.F_3 = 2$. If $F_4.(F_1 + F_2) = 4$ then $F_1.F_4 = F_2.F_4 = 2$.*

Proof. By symmetry and Lemma 4.2 we can assume, to get a contradiction, that $F_1.F_4 = 1$ and $F_2.F_4 = 3$. Then $(F_2 + F_4)^2 = 6$ and $\phi(F_2 + F_4) = 2$ whence, by Lemma 4.2, we can write $F_2 + F_4 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i.A_j = 1$ for $i \neq j$. But this gives the contradiction $8 = (F_2 + F_4).(F_1 + F_2 + F_3) \geq 3\phi(F_1 + F_2 + F_3) = 9$. □

Lemma 4.4. *Let $L > 0$ be a line bundle on an Enriques surface with $L^2 \geq 0$. Then there exist (not necessarily distinct) divisors $F_i > 0, 1 \leq i \leq m$, such that $F_i^2 = 0$ and $L \sim F_1 + \ldots + F_m$.*

Definition 4.5. *We call such a decomposition of L an* **arithmetic genus 1 decomposition**.

Proof of Lemma 4.4. The assertion being clear for $L^2 = 0$ we suppose $L^2 > 0$. By Lemma 4.2 there is an $F_1 > 0$ such that $F_1.L = \phi(L)$. Since $\phi(L) \leq \lfloor\sqrt{L^2}\rfloor$, we have $(L - F_1)^2 \geq 0$. Again by Lemma 4.2 we have $L - F_1 > 0$ and $(L - F_1)^2 < L^2$ so that we can proceed by induction. □

Definition 4.6. *Let $L \geq 0$ be an effective line bundle on an Enriques surface with $L^2 \geq 0$. Then L is said to be of* **small type** *if either $L = 0$ or for every decomposition of L with*

$$L \equiv a_1E_1 + \ldots + a_rE_r, \quad E_i > 0, \quad E_i^2 = 0, \quad E_i.E_j > 0 \text{ for } i \neq j,$$

and $a_i > 0$, we have that $a_i = 1$ for all $1 \leq i \leq r$.

The next two results are immediate consequences of the previous ones.

Lemma 4.7. *Let $L \geq 0$ be an effective line bundle on an Enriques surface with $L^2 \geq 0$. Then L is of small type if and only if either* (i) *$L^2 = 0$ and L is either trivial or primitive, or* (ii) *$L^2 > 0$ and $(L - 2F)^2 < 0$ for any $F > 0$ with $F^2 = 0$ and $F.L = \phi(L)$.*

Lemma 4.8. *Let $L \geq 0$ be an effective line bundle on an Enriques surface with $L^2 \geq 0$. Then L is of small type if and only if it is of one of the following types (where $E_i > 0$, $E_i^2 = 0$ and E_i primitive): (a) $L = 0$; (b) $L^2 = 0$, $L \sim E_1$; (c) $L^2 = 2$, $L \sim E_1 + E_2$, $E_1.E_2 = 1$; (d) $L^2 = 4$, $\phi(L) = 2$, $L \sim E_1 + E_2$, $E_1.E_2 = 2$; (e) $L^2 = 6$, $\phi(L) = 2$, $L \sim E_1 + E_2 + E_3$, $E_1.E_2 = E_1.E_3 = E_2.E_3 = 1$; (f) $L^2 = 10$, $\phi(L) = 3$, $L \sim E_1 + E_2 + E_3$, $E_1.E_2 = 1$, $E_1.E_3 = E_2.E_3 = 2$.*

Given an effective line bundle L with $L^2 > 0$, among all arithmetic genus 1 decompositions of L we want to choose the most convenient for us (in a sense that will be clear in the following sections). To this end let us first record the following

Lemma 4.9. *Let $L > 0$ be a line bundle on an Enriques surface such that $L^2 > 0$ and suppose there exists an $F > 0$ with $F^2 = 0, \phi(L) = F.L$ and $(L - 2F)^2 > 0$. Then there exist an integer $k \geq 2$ and an $F' > 0$ with $(F')^2 = 0, F'.F > 0, (L - kF)^2 > 0$ and $F'.(L - kF) = \phi(L - kF)$.*

Proof. As $(L-2F)^2 > 0$ we can choose an integer $k \geq 2$ such that $(L-kF)^2 > 0$ and $(L-(k+1)F)^2 \leq 0$. Set $L' = L - kF$, so that $L' > 0$ by Lemma 4.2 and pick any $F' > 0$ such that $(F')^2 = 0$ and $F'.L' = \phi(L')$. If $(L' - F')^2 > 0$ then $F' \not\equiv F$, whence $F'.F > 0$ by Lemma 4.2 and we are done.

If $(L' - F')^2 \leq 0$ one easily sees that $((L')^2, \phi(L')) = (2, 1)$ or $(4, 2)$ and $L' \sim F' + F''$ with $F'' > 0$, $(F'')^2 = 0$ and $F'.F'' = 1, 2$. Hence either $F'.F > 0$ or $F''.F > 0$ and we are again done. □

Now for any line bundle $L > 0$ which is *not of small type* with $L^2 > 0$ and $\phi(L) = F.L$ for some $F > 0$ with $F^2 = 0$, define

$$\alpha_F(L) = \min\{k \geq 2 \;\; | \;\; (L - kF)^2 \geq 0 \text{ and if } (L - kF)^2 > 0 \text{ there exists } F' > 0 \text{ with } (F')^2 = 0, F'.F > 0 \text{ and } F'.(L - kF) \leq \phi(L)\}. \tag{9}$$

By Lemma 4.9, $\alpha_F(L)$ exists and it is easily seen that an equivalent definition is

$$\alpha_F(L) = \min\{k \geq 2 \;\; | \;\; (L - kF)^2 \geq 0 \text{ and if } (L - kF)^2 > 0 \text{ there exists } F' > 0 \text{ with } (F')^2 = 0, F'.F > 0 \text{ and } F'.(L - kF) = \phi(L - kF)\}. \tag{10}$$

If $L^2 = 0$ and L is *not of small type*, then let $k \geq 2$ be the maximal integer such that there there exists an $F > 0$ with $F^2 = 0$ and $L \equiv kF$. In this case we define $\alpha_F(L) = k$.

Lemma 4.10. *Let $L > 0$ be a line bundle not of small type with $L^2 > 0$ and $(L^2, \phi(L)) \neq (16, 4)$, $(12, 3)$, $(8, 2)$, $(4, 1)$. Then $(L - \alpha_F(L)F)^2 > 0$.*

Proof. Set $\alpha = \alpha_F(L)$. Assume that $(L - \alpha F)^2 = 0$. Then, since $L^2 > 0$, we have $L \sim \alpha F + F'$ for some $F' > 0$ with $(F')^2 = 0$ and $F.F' = \phi(L)$. Now $(L - (\alpha - 1)F)^2 = (F + F')^2 = 2\phi(L) > 0$ and $F'.(L - (\alpha - 1)F) = \phi(L)$, whence $\alpha = 2$. Therefore $L \sim 2F + F'$, whence $L^2 = 4F.F' = 4\phi(L)$, which gives $\phi(L)^2 \leq 4\phi(L)$, in other words $\phi(L) \leq 4$ and we are done. □

Finally we recall a definition and some results, proved in [KL2] and [KL1], that will be used throughout the article.

Lemma 4.11. [KL2, Lemma2.4] *Let $L > 0$ and $\Delta > 0$ be divisors on an Enriques surface with $L^2 \geq 0$, $\Delta^2 = -2$ and $k := -\Delta.L > 0$. Then there exists an $A > 0$ such that $A^2 = L^2$, $A.\Delta = k$ and $L \sim A + k\Delta$. Moreover, if L is primitive, then so is A.*

Definition 4.12. *An effective line bundle L on a K3 or Enriques surface is said to be* **quasi-nef** *if $L^2 \geq 0$ and $L.\Delta \geq -1$ for every Δ such that $\Delta > 0$ and $\Delta^2 = -2$.*

Theorem 4.13. [KL1, Cor.2.5] *An effective line bundle L on a K3 or Enriques surface is quasi-nef if and only if $L^2 \geq 0$ and either $h^1(L) = 0$ or $L \equiv nE$ for some $n \geq 2$ and some primitive and nef divisor $E > 0$ with $E^2 = 0$.*

We will often make use of the following simple

Lemma 4.14. *Let L be a nef and big line bundle on an Enriques surface and let F be a divisor satisfying $F.L < 2\phi(L)$ (respectively $F.L = \phi(L)$ and L is ample). Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ we have $F^2 = 0$, $h^0(F) = 1$, $h^1(F) = 0$ and F is primitive and quasi-nef (resp. nef).*

Proof. If $h^0(F) \geq 2$ we can write $|F| = |M| + G$, with M the moving part and $G \geq 0$ the fixed part of $|F|$. By [CD, Prop.3.1.4] we get $F.L \geq 2\phi(L)$, a contradiction. Then $h^0(F) \leq 1$ and if $F > 0$ and $F^2 \geq 0$ it follows that $F^2 = 0$ and $h^1(F) = 0$ by Riemann-Roch. Hence F is quasi-nef and primitive by Theorem 4.13. If $F.L = \phi(L)$, L is ample and F is not nef, by Lemma 4.11 we can write $F \sim F_0 + \Gamma$ with $F_0 > 0$, $F_0^2 = 0$ and Γ a nodal curve. But then $F_0.L < \phi(L)$. □

5. Main results on extendability of Enriques surfaces

It is well-known that abelian and hyperelliptic surfaces are nonextendable (see for example [GLM, Rmk.3.12]). In the case of K3 surfaces the extendability problem is open, but beautiful answers are known for general K3's (even with assigned Picard lattice) ([CLM1, CLM2, Be]). Let us deal now with Enriques surfaces.

We will state here a simplification of Corollary 2.4 that will be a central ingredient for us. An analogous result can be written for K3 surfaces.

Proposition 5.1. *Let $S \subset \mathbb{P}^r$ be an Enriques surface and denote by H its hyperplane section. Suppose we can find a nef and big (whence > 0) line bundle D_0 on S with $\phi(D_0) \geq 2, H^1(H-D_0) = 0$ and such that the following conditions are satisfied by the general element $D \in |D_0|$:*

(i) *the Gaussian map Φ_{H_D,ω_D} is surjective;*
(ii) *the multiplication map μ_{V_D,ω_D} is surjective, where*
$V_D := \operatorname{Im}\{H^0(S, \mathcal{O}_S(H - D_0)) \to H^0(D, \mathcal{O}_D(H - D_0))\}$;
(iii) $h^0(\mathcal{O}_D(2D_0 - H)) \leq \frac{1}{2}D_0^2 - 2$.

Then S is nonextendable.

Proof. Note that $D_0^2 \geq \phi(D_0)^2 \geq 4$. Now the line bundle D_0 is base-point free since $\phi(D_0) \geq 2$ by [CD, Prop.3.1.6, 3.1.4 and Thm.4.4.1]. Therefore we just apply Corollary 2.4 and Remark 2.2. □

Our first observation will be that, for many line bundles H, a line bundle D_0 satisfying the conditions of Proposition 5.1 can be found with the help of Ramanujam's vanishing theorem.

Proposition 5.2. *Let $S \subset \mathbb{P}^r$ be an Enriques surface, denote by H its hyperplane section and assume that H is not 2-divisible in* $\operatorname{Num} S$. *Suppose there exists an effective line bundle B on S with the following properties:*

(i) $B^2 \geq 4$ *and* $\phi(B) \geq 2$,
(ii) $(H - 2B)^2 \geq 0$ *and* $H - 2B \geq 0$,
(iii) $H^2 \geq 64$ *if* $B^2 = 4$ *and* $H^2 \geq 54$ *if* $B^2 = 6$.

Then S is nonextendable.

Proof. We first claim that we can find a *nef* divisor $D' > 0$ still satisfying (i)-(iii) with $D' \leq B$, $(D')^2 = B^2$ and $\phi(D') = \phi(B)$ by using *Picard-Lefschetz reflections*.

Recall that if Γ is a nodal curve on an Enriques surface, then the Picard-Lefschetz reflection with respect to Γ acting on $\operatorname{Pic} S$ is defined as $\pi_\Gamma(L) := L + (L.\Gamma)\Gamma$. It is straightforward to check that $\pi_\Gamma(\pi_\Gamma(L)) = L$ and that $\pi_\Gamma(L).\pi_\Gamma(L') = L.L'$ for any $L, L' \in \operatorname{Pic} S$. Moreover π_Γ preserves effectiveness [BPV, Prop.VIII.16.3] and the function ϕ, when $L^2 > 0$.

Now if B is not nef, then there is a nodal curve Γ such that $\Gamma.B < 0$. By the properties of π_Γ just mentioned and the fact that clearly $0 < \pi_\Gamma(B) < B$, it follows that $\pi_\Gamma(B)^2 = B^2$ and $\phi(\pi_\Gamma(B)) = \phi(B)$, whence $\pi_\Gamma(B)$ still satisfies (i)-(iii). If $\pi_\Gamma(B)$ is not nef, we repeat the process, which must eventually end, as $\pi_\Gamma(B) < B$.

We have therefore found the desired nef divisor D'.

Since $H - D' \geq H - B > H - 2B \geq 0$ and $(D')^2 > 0$, we have $D'.(H - D') > 0$.

Now define the set

$$\Omega(D') = \{M \in \operatorname{Pic} S : M \geq D', M \text{ is nef, satisfies (i)-(ii) and } M.(H - M) \leq D'.(H - D')\}.$$

We have just seen that this set is nonempty.

Note that for any $M \in \Omega(D')$ we have $H - 2M > 0$, whence $H.M < \frac{1}{2}H^2$ is bounded. Let then D_0 be a *maximal* divisor in $\Omega(D')$, that is a divisor in $\Omega(D')$ such that $H.D_0 \geq H.M$ for any $M \in \Omega(D')$. We want to show that $h^1(H - 2D_0) = 0$.

Set $R = H - 2D_0$. Assume, to get a contradiction that $h^1(H - 2D_0) > 0$. Since $R^2 \geq 0$ it follows from Ramanujam vanishing [BPV, Cor.II.12.3] that $R + K_S$ is not 1-connected, whence $R + K_S \sim R_1 + R_2$, for $R_1 > 0$ and $R_2 > 0$ with $R_1.R_2 \leq 0$.

We can assume that $R_1.H \leq R_2.H$. Define $D_1 = D_0 + R_1$. If D_1 is nef, $\phi(D_1)$ is calculated by a nef divisor, whence $\phi(D_1) \geq \phi(D') \geq 2$ and $D_1^2 \geq D_0^2 \geq (D')^2 \geq 4$ (since $D_1 \geq D_0 \geq D'$). Moreover

$$(H - 2D_1)^2 = (R - 2R_1)^2 = (R_2 - R_1)^2 = R^2 - 4R_1.R_2 \geq R^2 \geq 0,$$

and since

$$(H - 2D_1).H = (R - 2R_1).H = (R_2 - R_1).H \geq 0,$$

we get by Riemann-Roch and the fact that H is not 2-divisible in $\operatorname{Num} S$, that $H - 2D_1 > 0$.

Furthermore

$$D_1.(H - D_1) = (D_0 + R_1).(H - D_0 - R_1) = D_0.(H - D_0) + R_1.R_2 \leq D_0.(H - D_0),$$

whence D_1 is an element of $\Omega(D')$ with $H.D_1 > H.D_0$, contrary to our assumption that D_0 is maximal.

Hence D_1 cannot be nef and there exists a nodal curve Γ with $\Gamma.D_1 < 0$ (whence $\Gamma.R_1 < 0$). Since H is ample we must have $\Gamma.(H - D_1) \geq -\Gamma.D_1 + 1 \geq 2$. Let now $D_2 = D_1 - \Gamma$. Since $\Gamma.R_1 < 0$ we have $D_2 \geq D_0$, whence, if D_2 is nef, we have as above that $\phi(D_2) \geq \phi(D') \geq 2$ and $D_2^2 \geq D_0^2 \geq (D')^2 \geq 4$. Moreover $H - 2D_2 > H - 2D_1 > 0$ and

$$(H - 2D_2)^2 = (H - 2D_1 + 2\Gamma)^2 = (H - 2D_1)^2 - 8 + 4(H - 2D_1).\Gamma \geq (H - 2D_1)^2 + 4 > 0.$$

Furthermore we also have

$$\begin{aligned} D_2.(H - D_2) &= (D_1 - \Gamma).(H - D_1 + \Gamma) = D_1.(H - D_1) - \Gamma.(H - D_1) + \Gamma.D_1 + 2 \leq \\ &\leq D_1.(H - D_1) - 1 < D_1.(H - D_1) \leq D_0.(H - D_0), \end{aligned}$$

whence $D_2 = D_0 + (R_1 - \Gamma) \neq D_0$. Now if D_2 is nef, then it is an element of $\Omega(D')$ with $H.D_2 > H.D_0$, contrary to our assumption that D_0 is maximal.

Hence D_2 cannot be nef, and we repeat the process by finding a nodal curve Γ_1 such that $\Gamma_1.(R_1 - \Gamma) < 0$ and so on. However, since R_1 can only contain finitely many nodal curves, the process must end, that is $h^1(H - 2D_0) = 0$, as we claimed.

Note that since $D_0^2 \geq (D')^2 = B^2$, then D_0 also satisfies (iii) above, that is $H^2 \geq 64$ if $D_0^2 = 4$ and $H^2 \geq 54$ if $D_0^2 = 6$. Furthermore D_0 is base-point free since it is nef with $\phi(D_0) \geq \phi(D') \geq 2$ [CD, Prop.3.1.6, 3.1.4 and Thm.4.4.1].

Now let D be a general smooth curve in $|D_0|$. We have

$$\deg(H - D_0)_{|D} = (H - D_0).D_0 = D_0^2 + (H - 2D_0).D_0 \geq 2g(D),$$

where we have used that $(H - 2D_0).D_0 \geq \phi(D_0) \geq 2$ by Lemma 4.4. Since D is not hyperelliptic, it follows that $(H - D_0)_{|D}$ is base-point free and birational, whence the map $\mu_{(H-D_0)_{|D},\omega_D}$ is surjective by [AS, Thm.1.6].

From the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(H - 2D_0) \longrightarrow \mathcal{O}_S(H - D_0) \longrightarrow \mathcal{O}_D(H - D_0) \longrightarrow 0,$$

and the fact that $h^1(\mathcal{O}_D(H - D_0)) = 0$ for reasons of degree, we find $h^1(H - D_0) = 0$.

Now to show that S is nonextendable, we only have left to show, by Proposition 5.1, that the map Φ_{H_D,ω_D} is surjective.

From $(H - 2D_0).D_0 \geq 2$ again, we get $\deg H_D \geq 4g(D) - 2$, whence by [BEL, Thm.2], the map Φ_{H_D,ω_D} is surjective provided that $\operatorname{Cliff}(D) \geq 2$. This is satisfied if $D_0^2 \geq 8$ by [KL2, Cor.1 and Prop.4.15].

If $D_0^2 = 6$, then $g(D) = 4$, whence by [Wa, Prop.1.10], the map Φ_{H_D,ω_D} is surjective if $h^0(\mathcal{O}_D(3D_0 + K_S - H)) = 0$ (see also Theorem 5.3(b) below). Since $H^2 \geq 54$, we get by the Hodge index theorem that $H.D \geq 18$ with equality if and only if $H \equiv 3D_0$. If $H.D_0 > 18$, we get $\deg \mathcal{O}_D(3D_0 + K_S - H) < 0$ and Φ_{H_D,ω_D} is surjective. If $H \equiv 3D_0$, then either $H \sim 3D_0$ and $h^0(\mathcal{O}_D(3D_0 + K_S - H)) = h^0(\mathcal{O}_D(K_S)) = 0$ or $H \sim 3D_0 + K_S$ and then we exchange D_0 with $D_0 + K_S$ and we are done.

If $D_0^2 = 4$, then $g(D) = 3$, whence by [Wa, Prop.1.10], the map Φ_{H_D,ω_D} is surjective if $h^0(\mathcal{O}_D(4D_0 - H)) = 0$ (see also Theorem 5.3(a) below). Since $H^2 \geq 64$, we get by the Hodge index theorem that $H.D \geq 17$, whence $\deg \mathcal{O}_D(4D_0 - H) < 0$ and the map Φ_{H_D,ω_D} is surjective. $\square$

We recall the following result on Gaussian maps on curves on Enriques surfaces.

Theorem 5.3. [KL3] *Let S be an Enriques surface, let L be a very ample line bundle on S and let D_0 be a line bundle such that D_0 is nef, $D_0^2 \geq 4, \phi(D_0) \geq 2$ and $H^1(D_0 - L) = 0$. Let D be a general divisor in $|D_0|$. Then the Gaussian map $\Phi_{L_{|D},\omega_D}$ is surjective if one of the hypotheses below is satisfied:*

(a) $D_0^2 = 4$ *and* $h^0(\mathcal{O}_D(4D_0 - L)) = 0$*;*
(b) $D_0^2 = 6$ *and* $h^0(\mathcal{O}_D(3D_0 + K_S - L)) = 0$*;*
(c) $D_0^2 \geq 8$ *and* $h^0(2D_0 - L) = 0$*;*
(d) $D_0^2 \geq 12$ *and* $h^0(2D_0 - L) = 1$*;*
(e) $H^1(L_{|D}) = 0$*,* $L.D_0 \geq \frac{1}{2}D_0^2 + 2 \geq 6$ *and* $h^0(2D_0 - L) \leq \operatorname{Cliff}(D) - 2$.

We now get an improvement of Proposition 5.2 in the cases $B^2 = 6$ and $B^2 = 4$.

Proposition 5.4. *Let $S \subset \mathbb{P}^r$ be an Enriques surface, denote by H its hyperplane section and assume that H is not 2-divisible in* $\operatorname{Num} S$*. Suppose there exists a line bundle B on S with the following properties:*

(i) $B^2 = 6$ *and* $\phi(B) = 2$*,*
(ii) $(H - 2B)^2 \geq 0$ *and* $H - 2B \geq 0$*,*
(iii) $h^0(3B - H) = 0$ *or* $h^0(3B + K_S - H) = 0$*.*

Then S is nonextendable.

Proof. Argue exactly as in the proof of Proposition 5.2 and let D', D_0 and D be as in that proof, so that, in particular, $D_0^2 \geq (D')^2 = 6$. If $D_0^2 \geq 8$, we are done by Proposition 5.2. If $D_0^2 = 6$ write $D_0 = D' + M$ with $M \geq 0$. Since both D_0 and D' are nef we find $6 = D_0^2 = (D')^2 + D'.M + D_0.M \geq 6$, whence $D'.M = D_0.M = 0$, so that $M^2 = 0$. Therefore $M = 0$ and $D_0 = D'$, whence $3D_0 - H \sim 3D' - H \leq 3B - H$. It follows that either $h^0(3D_0 - H) = 0$ or $h^0(3D_0 + K_S - H) = 0$. Possibly after exchanging D_0 with $D_0 + K_S$, we can assume that $h^0(3D_0 + K_S - H) = 0$. From the exact sequence

$$0 \longrightarrow \mathcal{O}_S(2D_0 + K_S - H) \longrightarrow \mathcal{O}_S(3D_0 + K_S - H) \longrightarrow \mathcal{O}_D(3D_0 + K_S - H) \longrightarrow 0,$$

and the fact that $h^1(2D_0 + K_S - H) = h^1(H - 2D_0) = 0$, we get $h^0(\mathcal{O}_D(3D_0 + K_S - H)) = 0$, whence the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(b).

The multiplication map μ_{V_D,ω_D} is surjective as in the proof of Proposition 5.2, whence S is nonextendable by Proposition 5.1. □

Proposition 5.5. *Let $S \subset \mathbb{P}^r$ be an Enriques surface, denote by H its hyperplane section and assume that H is not 2-divisible in* $\operatorname{Num} S$*. Suppose there exists a line bundle B on S with the following properties:*

(i) *B is nef,* $B^2 = 4$ *and* $\phi(B) = 2$*,*
(ii) $(H - 2B)^2 \geq 0$ *and* $H - 2B \geq 0$*,*
(iii) $H.B > 16$*.*

Then S is nonextendable.

Proof. Argue exactly as in the proof of Proposition 5.2 and let D', D_0 and D be as in that proof. Since B is assumed to be nef, we have $D' = B$, and since $D_0 \geq D'$, we get $H.D_0 > 16$. If $D_0^2 \geq 8$, we are done by Proposition 5.2. If $D_0^2 = 6$, then we must have $D_0 > D' = B$, so that $H.D_0 \geq 18$ whence $(3D_0 - H).D_0 \leq 0$. This gives that if $3D_0 - H > 0$, then it is a nodal cycle, whence either $h^0(3D_0 - H) = 0$ or $h^0(3D_0 + K_S - H) = 0$. Now we are done by Proposition 5.4.

If $D_0^2 = 4$, then, as in the proof of Proposition 5.4, $D_0 = D' = B$, whence the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(a), since $\deg \mathcal{O}_D(4D_0 - H) < 0$. The multiplication map μ_{V_D,ω_D} is surjective as in the proof of Proposition 5.2, whence S is nonextendable by Proposition 5.1. □

In several cases the following will be very useful:

Lemma 5.6. *Let $S \subset \mathbb{P}^r$ be an Enriques surface with hyperplane section $H \sim 2B + A$, for B nef, $B^2 \geq 2$, $A^2 = 0$, $A > 0$ primitive, $H^2 \geq 28$ and satisfying one of the following conditions:*

(i) *A is quasi-nef and $(B^2, A.B) \notin \{(4,3),(6,2)\}$;*
(ii) *$\phi(B) \geq 2$ and $(B^2, A.B) \notin \{(4,3),(6,2)\}$;*
(iii) *$\phi(B) = 1$, $B^2 = 2l$, $B \sim lF_1 + F_2$, $l \geq 1$, $F_i > 0$, $F_i^2 = 0$, $i = 1,2$, $F_1.F_2 = 1$, and either*
  (a) *$l \geq 2$, $F_i.A \leq 3$ for $i = 1,2$ and $(l, F_1.A, F_2.A) \neq (2,1,1)$; or*
  (b) *$l = 1$, $5 \leq B.A \leq 8$, $F_i.A \geq 2$ for $i = 1,2$ and $(\phi(H), F_1.A, F_2.A) \neq (6,4,4)$.*

Then S is nonextendable.

Proof. Note that possibly after replacing B with $B+K_S$ if $B^2 = 2$ we can, without loss of generality, assume that B is base-component free.

We first prove the lemma under hypothesis (i).

We have that $B + A$ is nef, since any nodal curve Γ with $\Gamma.(B + A) < 0$ would have to satisfy $\Gamma.A = -1$ and $\Gamma.B = 0$, whence $\Gamma.H = -1$, a contradiction.

Now let $D_0 = B + A$. Then $D_0^2 = B^2 + 2B.A \geq 6$, since $A.B \geq 2$ for $2A.B = A.H \geq \phi(H) \geq 3$, and $\phi(D_0) \geq \phi(B) \geq 1$.

If $\phi(D_0) = 1 = F.D_0$ for some $F > 0$ with $F^2 = 0$ we get $F.B = 1$, $F.A = 0$ giving the contradiction $F.H = 2$. Therefore $\phi(D_0) \geq 2$.

One easily checks that (i) implies $D_0^2 \geq 12$. Since $h^0(2D_0 - H) = h^0(A) = 1$ by Theorem 4.13, we have that Φ_{H_D,ω_D} is surjective by Theorem 5.3(d).

Also $h^1(H - 2D_0) = h^1(-A) = 0$, again by Theorem 4.13, so that we have $V_D = H^0(\mathcal{O}_D(H - D_0))$. As $H - D_0 = B$ is base-component free and $|D_0|$ is base-point free and birational by [CD, Lemma4.6.2, Thm.4.6.3 and Prop.4.7.1], also V_D is base-point free and is either a complete pencil or birational, and then μ_{V_D,ω_D} is surjective by the base-point free pencil trick and by [AS, Thm.1.6] (see also (14)). Then S is nonextendable by Proposition 5.1.

Therefore the lemma is proved under the assumption (i) and, in particular, the whole lemma is proved with the additional assumption that A is quasi-nef.

Now assume that A is not quasi-nef. Then there is a $\Delta > 0$ with $\Delta^2 = -2$ and $\Delta.A \leq -2$. We have $\Delta.B \geq 2$ by the ampleness of H. Furthermore, among all such Δ's we will choose a minimal one, that is such that no $0 < \Delta' < \Delta$ satisfies $(\Delta')^2 = -2$ and $\Delta'.A \leq -2$.

We now claim that $B_0 := B + \Delta$ is nef. Indeed, if there is a nodal curve Γ with $\Gamma.(B + \Delta) < 0$ then $\Gamma.\Delta < 0$ and we must have $\Delta_1 := \Delta - \Gamma > 0$ with $\Delta_1^2 = -4 - 2\Delta.\Gamma$.

If $\Delta.\Gamma \leq -2$ then $\Delta_1^2 \geq 0$ whence $\Delta_1.A \geq 0$ by Lemma 4.2 and $\Gamma.A \leq \Delta.A \leq -2$, contradicting the minimality of Δ. Therefore $\Delta.\Gamma = -1$, $\Gamma.B = 0$ and $\Delta_1^2 = -2$. The ampleness of H implies $\Gamma.A > 0$, whence $\Delta_1.A < \Delta.A \leq -2$, again contradicting the minimality of Δ.

Therefore $B_0 := B + \Delta$ is nef with $B_0^2 \geq 2 + B^2$, and, as $\phi(B_0)$ is computed by a nef isotropic divisor, we have that $\phi(B_0) \geq \phi(B)$.

We also note that $H - 2B_0 \sim A - 2\Delta > 0$ and primitive by Lemma 4.11 with $(H - 2B_0)^2 \geq 0$.

Under the assumptions (ii), we have $\phi(B_0) \geq 2$. Then S is nonextendable by Proposition 5.2 if $B_0^2 \geq 8$. If $B_0^2 = 6$, we have $B^2 = 4$ and $\Delta.B = 2$, so that $\Delta.A = -2$ or -3 by the ampleness of H. Hence $H \sim 2B_0 + A'$, with $B_0^2 = 6$ and $A' \sim A - 2\Delta$ satisfies $(A')^2 = 0$ or 4. In the first case we are done by conditions (i) if A' is quasi-nef (because $B_0.A' = (B + \Delta).A' \geq 4$), and if not we can just repeat the process and find that S is nonextendable by Proposition 5.2 (because we find a divisor B_0' with $(B_0')^2 \geq 8$). In the case $(A')^2 = 4$ we have $A'.B_0 \geq 5$ by the Hodge index theorem. Therefore $(3B_0 - H).B_0 = (B_0 - A').B_0 \leq 1 < \phi(B_0)$, so that if $3B_0 - H > 0$ it is a nodal cycle. Hence either $h^0(3B_0 - H) = 0$ or $h^0(3B_0 + K_S - H) = 0$ and S is nonextendable by Proposition 5.4.

We have therefore shown that S is nonextendable under conditions (ii).

Now assume (iii) and, using Lemma 4.11, write $A \sim A_0 + k\Delta$ with $A_0 > 0$ primitive, $A_0^2 = 0$ and $k := -\Delta.A = \Delta.A_0 \geq 2$.

As $F_i.A = F_i.A_0 + kF_i.\Delta$, the primitivity of F_i, $\Delta.B \geq 2$ and the hypotheses in case (iii-a) yield the only possibility $k = 2$, $F_1.\Delta = F_1.A_0 = 1$. Then $H \sim 2B_0 + A_0$ with $B_0^2 \geq 6$, $\phi(B_0) \geq 2$, B_0 nef, $A_0^2 = 0$ and $B_0.A_0 = (B + \Delta).A_0 \geq 3$, so that conditions (ii) are satisfied and S is nonextendable.

Finally we assume we are in case (iii-b), so that $F_i.A \leq 6$ for $i = 1, 2$ by hypothesis.

Suppose $\Delta.F_1 \leq 0$. Then $F_2.\Delta \geq 2$. As $6 \geq F_2.A = F_2.A_0 + kF_2.\Delta$, we must have $k = F_2.\Delta = 2$, so that $\Delta.F_1 = 0$ and $4 \leq F_2.A \leq 6$. In particular, $F_1.B_0 = 1$, so that $B_0 \sim 2F_1 + F_2'$, where $F_2' \sim F_2 + \Delta - F_1 > 0$ satisfies $(F_2')^2 = 0$. We have $F_1.A_0 = F_1.A \leq 4$, and equality implies $F_2.A = 4$, whence $F_2 \equiv A_0$, so that $F_1.A_0 = F_1.F_2 = 1$, a contradiction. Hence $F_1.A_0 \leq 3$. Moreover $F_2'.A_0 = (F_2 + \Delta - F_1).A_0 = (F_2 - F_1).A - 2 \leq 2$, as $F_2.A \leq 6$. Also it cannot be $(F_1.A_0, F_2'.A_0) = (1, 1)$, for then $F_1.A = 1$. Therefore $H \sim 2B_0 + A_0$ satisfies the conditions in (iii-a), so that S is nonextendable.

We can therefore assume $\Delta.F_1 > 0$, and by symmetry, also $\Delta.F_2 > 0$. Hence $\phi(B_0) \geq 2$.

If $k \geq 3$, then $F_i.A = F_i.A_0 + kF_i.\Delta \geq 4$ for $i = 1, 2$, so that by our assumptions we can only have $k = 3$, $F_i.A = 4$ and $F_i.\Delta = F_i.A_0 = 1$. Then $B.A = 8$ and $H^2 = 40$, so that $\phi(H) \leq 5$ by hypothesis. Pick any isotropic divisor $F > 0$ satisfying $F.H = \phi(H)$. Since $(A')^2 = 4$ we have $5 \geq F.H = 2F.B_0 + F.A' \geq 5$, so that $F.H = 5$, $F.A' = 1$, $(A' - 2F)^2 = 0$, $A' - 2F > 0$, and $(A' - 2F).H = (A - 2\Delta - 2F).H = 4$, a contradiction.

Therefore $k = 2$, so that $A_0^2 = 0$. As $B_0.A_0 = (B + \Delta).A_0 = B.A_0 + 2 \geq 3$, we see that the conditions (ii) are satisfied, unless possibly if $B_0^2 = 4$ and $B.A_0 = 1$. In this case $B.\Delta = 2$ and $A_0 \equiv F_i$, for $i = 1$ or 2. Hence $\Delta.B = \Delta.(F_1 + F_2) = 3$, a contradiction. Therefore the conditions (ii) are satisfied and S is nonextendable. $\square$

We also have the following helpful tools to check the surjectivity of μ_{V_D,ω_D} in the cases where $h^1(H - 2D_0) \neq 0$. The first lemma holds on any smooth surface.

Lemma 5.7. *Let S be a smooth surface, let L be a line bundle on S and let D_1 and D_2 be two effective nonzero divisors on S not intersecting the base locus of $|L|$ and such that $h^0(\mathcal{O}_{D_1}) = 1$ and $h^0(\mathcal{O}_{D_1}(-L)) = h^0(\mathcal{O}_{D_2}(-D_1)) = 0$. For any divisor $B > 0$ on S define $V_B = \operatorname{Im}\{H^0(S, L) \to H^0(B, \mathcal{O}_B(L))\}$. If the multiplication maps $\mu_{V_{D_1},\omega_{D_1}}$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ are surjective then μ_{V_D,ω_D} is surjective for general $D \in |D_1 + D_2|$.*

Proof. Let $D' = D_1 + D_2$. By hypothesis we have the exact sequence

$$0 \longrightarrow H^0(\omega_{D_1}) \longrightarrow H^0(\omega_{D'}) \xrightarrow{\psi} H^0(\omega_{D_2}(D_1)) \longrightarrow 0.$$

Moreover by definition we have two surjective maps $\pi_i : V_{D'} \to V_{D_i}, i = 1, 2$, whence a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d, "\varphi"] & V_{D'} \otimes H^0(\omega_{D'}) \arrow[r, "\pi_2 \otimes \psi"] \arrow[d, "\mu_{V_{D'},\omega_{D'}}"] & V_{D_2} \otimes H^0(\omega_{D_2}(D_1)) \arrow[r] \arrow[d, "\mu_{V_{D_2},\omega_{D_2}(D_1)}"] & 0 \\
0 \arrow[r] & H^0(\omega_{D_1}(L)) \arrow[r, "\chi"] & H^0(\omega_{D'}(L)) \arrow[r] & H^0(\omega_{D_2}(D_1 + L)) \arrow[r] & 0
\end{tikzcd}$$

where $W := \operatorname{Ker} \pi_2 \otimes H^0(\omega_{D'}) + V_{D'} \otimes \operatorname{Ker} \psi$ and φ is just the restriction of $\mu_{V_{D'},\omega_{D'}}$ to this subspace. Since $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective, to conclude the surjectivity of $\mu_{V_{D'},\omega_{D'}}$ we just show the surjectivity of φ. Now the commutative diagram

$$\begin{tikzcd}
V_{D'} \otimes H^0(\omega_{D_1}) \arrow[r, "\cong"] \arrow[dr, "\pi_1 \otimes \mathrm{Id}"] & V_{D'} \otimes \operatorname{Ker} \psi \arrow[r] & H^0(\omega_{D'}(L)) \\
 & V_{D_1} \otimes H^0(\omega_{D_1}) \arrow[r, "\mu_{V_{D_1},\omega_{D_1}}"] & H^0(\omega_{D_1}(L)) \arrow[u, "\chi"]
\end{tikzcd}$$

and the injectivity of χ show that $H^0(\omega_{D_1}(L)) = \operatorname{Im}\mu_{V_{D_1},\omega_{D_1}} = \operatorname{Im}\varphi_{|V_{D'}\otimes\operatorname{Ker}\psi}$, as required.

Therefore $\mu_{V_{D'},\omega_{D'}}$ is surjective. Now for any divisor $B \in |D'|$, let $M_B = \operatorname{Ker}\{V_B \otimes \mathcal{O}_B \to L_{|B}\}$. By hypothesis we have that $V_{D'}$ globally generates $L_{|D'}$, whence we have an exact sequence

$$0 \longrightarrow M_{D'} \otimes \omega_{D'} \longrightarrow V_{D'} \otimes \omega_{D'} \to L_{|D'} \otimes \omega_{D'} \longrightarrow 0.$$

Since $h^1(\omega_{D_1}(L)) = h^1(\omega_{D_2}(D_1 + L)) = 0$ we get $h^1(L_{|D'} \otimes \omega_{D'}) = 0$ and the surjectivity of $\mu_{V_{D'},\omega_{D'}}$ implies that $h^1(M_{D'} \otimes \omega_{D'}) \leq \dim V_{D'} = \dim V_D$. By semicontinuity the same holds for a general $D \in |D'|$, whence μ_{V_D,ω_D} is surjective. □

Lemma 5.8. *Let S be an Enriques surface, let L be a very ample line bundle on S and let D_0 be a nef and big divisor on S such that $\phi(D_0) \geq 2$. Let $E > 0$ be such that $E^2 = 0$, $E.L = \phi(L)$ and define, on a general $D \in |D_0|$, $V_D = \operatorname{Im}\{H^0(\mathcal{O}_S(L - D_0)) \to H^0(\mathcal{O}_D(L - D_0))\}$.*

If $|L - D_0 - 2E|$ is base-component free, $h^1(D_0 + K_S - 2E) = h^2(D_0 + K_S - 4E) = 0$ and

$$h^0(L - 2D_0 - 2E) + h^0(\mathcal{O}_D(L - D_0 - 4E)) \leq \frac{1}{2}(L - D_0 - 2E)^2 - 1 \tag{11}$$

then μ_{V_D,ω_D} is surjective.

Proof. Consider the natural restriction maps $p_D : H^0(\mathcal{O}_S(L - D_0)) \to H^0(\mathcal{O}_D(L - D_0))$, $p'_D : H^0(\mathcal{O}_S(D_0+K_S)) \to H^0(\omega_D)$, $r_D : H^0(\mathcal{O}_S(L-D_0-2E)) \to H^0(\mathcal{O}_D(L-D_0-2E))$, $r'_D : H^0(\mathcal{O}_S(2E+D_0 + K_S)) \to H^0(\omega_D(2E))$. Then $V_D = \operatorname{Im} p_D$, $W_D := \operatorname{Im} r_D$ and let $\mu = \mu_{2E,D_0+K_S}$, $\mu' = \mu_{2E,L-D_0-2E}$ be the multiplication maps of line bundles on S. We have a commutative diagram

$$\begin{array}{ccc}
H^0(2E) \otimes H^0(L - D_0 - 2E) \otimes H^0(D_0 + K_S) & \xrightarrow{\operatorname{Id}\otimes\mu} & H^0(L - D_0 - 2E) \otimes H^0(2E + D_0 + K_S) \\
\big\downarrow{\scriptstyle \mu'\otimes\operatorname{Id}} & & \big\downarrow{\scriptstyle r_D\otimes r'_D} \\
H^0(L - D_0) \otimes H^0(D_0 + K_S) & & W_D \otimes H^0(\omega_D(2E)) \\
\big\downarrow{\scriptstyle p_D\otimes p'_D} & & \big\downarrow{\scriptstyle \mu_{W_D,\omega_D(2E)}} \\
V_D \otimes H^0(\omega_D) & \xrightarrow{\mu_{V_D,\omega_D}} & H^0(\mathcal{O}_D(L + K_S)).
\end{array}$$

Since $H^1(D_0 + K_S - 2E) = H^2(D_0 + K_S - 4E) = 0$ we have that μ is surjective by Castelnuovo-Mumford's lemma. At the same time, since $h^1(2E + K_S) = 0$, we have that r'_D is surjective. To conclude the surjectivity of μ_{V_D,ω_D}, by the above diagram, we just need to prove that $\mu_{W_D,\omega_D(2E)}$ is surjective. To see the latter note that, as D is general, W_D is base-point free and $\dim W_D - 2 = h^0(L - D_0 - 2E) - h^0(L - 2D_0 - 2E) - 2$, whence $\mu_{W_D,\omega_D(2E)}$ is surjective by the H^0-lemma [Gr, Thm.4.e.1] as soon as $h^1(\omega_D(2E - (L - D_0 - 2E)) \leq h^0(L - D_0 - 2E) - h^0(L - 2D_0 - 2E) - 2$, which is equivalent to (11) by Riemann-Roch on S and Serre duality on D. □

6. Strategy of the proof of Theorem 1.5

In this section we prove Theorem 1.5 for all very ample line bundles on an Enriques surface except for some concrete cases, and then we give the main strategy of the proof in these remaining cases, which will then be carried out in Sections 7-14. We also set some notation and conventions that will be used throughout the proofs, often without further mentioning.

Let $S \subset \mathbb{P}^r$ be an Enriques surface of sectional genus g and let H be its hyperplane bundle. As we will prove a result also for $g = 15$ and 17 (Proposition 15.1) we will henceforth assume $g \geq 17$ or $g = 15$, so that $H^2 = 2g - 2 \geq 32$ or $H^2 = 28$, and, as H is very ample, $\phi(H) \geq 3$. We choose a genus one pencil $|2E|$ such that $E.H = \phi(H)$ and, as H is not of small type by Lemma 4.8, we define $\alpha := \alpha_E(H)$ and $L_1 := H - \alpha E$, where $\alpha_E(H)$ is as in (9) and (10). By Lemmas 4.2 and 4.10 we

have that $L_1 > 0$ and $L_1^2 > 0$. Now suppose that L_1 is not of small type. Starting with $L_0 := H$ and $E_0 := E$ we continue the process inductively until we reach a line bundle of small type, as follows. Suppose given, for $i \geq 1$, $L_i > 0$ not of small type with $L_i^2 > 0$. We choose $E_i > 0$ such that $E_i^2 = 0$, $E_i.E_{i-1} > 0$, $E_i.L_i = \phi(L_i)$ and define $\alpha_i = \alpha_{E_i}(L_i)$ and $L_{i+1} = L_i - \alpha_i E_i$. Note that $L_{i+1} > 0$ by Lemma 4.2. Now if $L_{i+1}^2 = 0$ we write $L_{i+1} \equiv \alpha_{i+1} E_{i+1}$ and define $L_{i+2} = 0$, which is of small type by definition and we also have $E_{i+1}.E_i > 0$ because $L_i^2 > 0$. If $L_{i+1}^2 > 0$ then either L_{i+1} is of small type or we can continue.

We then get, for some integer $n \geq 1$:

$$H = \alpha E + \alpha_1 E_1 + \ldots + \alpha_{n-1} E_{n-1} + L_n, \tag{12}$$

with $\alpha \geq 2$, $\alpha_i \geq 2$ for $1 \leq i \leq n-1$ and L_n is of small type.

Moreover $E.E_1 \geq 1$, $E_i.E_{i+1} \geq 1$, E and E_i are primitive for all i, $L_i^2 > 0$ and $E_i.L_i = \phi(L_i)$ for $0 \leq i \leq n-2$ and $L_{n-1}^2 \geq 0$.

We record for later the following fact, which follows immediately from the definitions:

$$E_1.(H - \alpha E) \leq \phi(H) \text{ and if } \alpha \geq 3 \text{ then } E_1.(H - \alpha E) \geq \phi(H) + 1 - E.E_1. \tag{13}$$

Furthermore we claim that $\alpha_i = 2$ for $1 \leq i \leq n-1$. To see this we proceed by induction on i. If $(L_1 - 2E_1)^2 = 0$ then $\alpha_1 = 2$ by definition. On the other hand if $(L_1 - 2E_1)^2 > 0$ to see that $\alpha_1 = 2$ we just need to prove that $E_0.(L_1 - 2E_1) \leq \phi(L_1)$, or, equivalently, that $\phi(L_0) \leq E_1.L_0 + (2 - \alpha_0)E_1.E_0$. Now the latter holds both when $\alpha_0 = 2$ and, by (13), when $\alpha_0 \geq 3$. By induction and the same proof for $i = 1$ we can prove that $\alpha_i = 2$ for $1 \leq i \leq n-2$ and also for $i = n-1$ if $L_{n-1}^2 > 0$. Finally if $L_{n-1}^2 = 0$ we have $L_{n-2} \equiv 2E_{n-2} + \alpha_{n-1}E_{n-1}$, whence $(\alpha_{n-1}E_{n-2}.E_{n-1})^2 = \phi(L_{n-2})^2 \leq L_{n-2}^2 = 4\alpha_{n-1}E_{n-2}.E_{n-1}$. Therefore $\alpha_{n-1}E_{n-2}.E_{n-1} \leq 4$ and if $\alpha_{n-1} \geq 3$ we get $E_{n-2}.E_{n-1} = 1$, giving the contradiction $\alpha_{n-1} = \phi(L_{n-2}) \leq E_{n-1}.L_{n-2} = 2$ and the claim is proved.

We now search for a divisor B as in Proposition 5.2 to show that $S \subset \mathbb{P}^r$ is nonextendable.

Assume for the moment that H is not 2-divisible in $\operatorname{Num} S$ and that $n \geq 2$ (that is L_1 is not of small type).

If $n \geq 4$, then set $B = E + E_1 + E_2 + E_3$. We have $B^2 \geq 6$ with equality if and only if $E.E_2 = E.E_3 = E_1.E_3 = 0$. But the latter implies the contradiction $E_2 \equiv E \equiv E_3$. Hence $B^2 \geq 8$ and clearly $\phi(B) \geq 2$. Now

$$H - 2B = (\alpha - 2)E + 2\sum_{i=4}^{n-1} E_i + L_n \geq 0,$$

where the sum is empty if $n = 4$. Hence $(H - 2B)^2 \geq 0$, therefore B satisfies the conditions in Proposition 5.2 and S is nonextendable.

If $n = 3$, then $H = \alpha E + 2E_1 + 2E_2 + L_3$. Set $B = \lfloor \frac{\alpha}{2} \rfloor E + E_1 + E_2$. Then B satisfies the conditions in Proposition 5.2, whence S is nonextendable, unless

(I-A) $n = 3, E_2 \equiv E$, $E.E_1 = 1$.

(II) $n = 3, E.E_1 = E.E_2 = E_1.E_2 = 1$, $\alpha \in \{2, 3\}$, $H^2 \leq 52$.

If $n = 2$, then $H = \alpha E + 2E_1 + L_2$. Set $B = \lfloor \frac{\alpha}{2} \rfloor E + E_1$. Then B satisfies the conditions in Proposition 5.2, whence S is nonextendable, unless

(I-B) $n = 2, E.E_1 = 1$.

(III) $n = 2, E.E_1 = 2$, $\alpha \in \{2, 3\}$, $H^2 \leq 62$,

or $n = 2, E.E_1 = 3$, $\alpha \in \{2, 3\}$ and $H^2 \leq 52$. But the latter case does not occur. Indeed then $6 \leq 6 + E.L_2 = E.H = \phi(H) \leq 6$ by [KL2, Prop.1], whence $E.L_2 = 0$, therefore either $L_2 = 0$ or $L_2 \equiv E$. Now since $(E + E_1)^2 = 6$ and $\phi(E + E_1) = 2$, as E and E_1 are primitive, we can write

$E+E_1 \sim A_1+A_2+A_3$ with $A_i > 0$, $A_i^2 = 0$. Therefore $18 \geq 6+3\alpha+E_1.L_2 = (E+E_1).H \geq 3\phi(H) = 18$, whence $\alpha = 3$ and $E_1.H = 12$. But then $E_1.(H-2E) = 6$ so that $\alpha = 2$, a contradiction.

Now $L_n \geq 0$ and $L_n^2 \geq 0$ so that, if $L_n > 0$, it has (several) arithmetic genus 1 decompositions. We want to extract from them any divisors numerically equivalent to E or to E_1, if possible. If, for example, we give priority to E, we will write $L_n \equiv E + L'_n$ and then, if L'_n has an arithmetic genus 1 decomposition with E_1 present, we write $L'_n \equiv E_1 + M_n$. In case the priority is given to E_1 we do it first with E_1 and then with E. Finally, for a reason that will be clear below, in the case (I-A), where only M_3 is defined, we will set $M_2 = M_3$.

To avoid treating the same case more times we make the following choice of **"removing conventions"**:

(I-A) Remove E and E_1 from L_3, the one with lowest intersection number with L_3 first, giving priority to E_1 in case $E.L_3 = E_1.L_3$.
(I-B) Remove E and E_1 from L_2, the one with lowest intersection number with L_2 first, giving priority to E in case $E.L_2 = E_1.L_2$.
(II) Remove E, E_1 and E_2 from L_3, the one with lowest intersection number with L_3 first, giving priority to E first and then to E_2.
(III) Remove E and E_1 from L_2, the one with lowest intersection number with L_2 first, giving priority to E in case $E.L_2 = E_1.L_2$.

At the end we get the following cases where the extendability of S still has to be checked, where $\gamma, \delta \in \{2,3\}$:

(I) $H \equiv \beta E + \gamma E_1 + M_2$, $E.E_1 = 1$, $H^2 \geq 32$ or $H^2 = 28$,
(II) $H \equiv \beta E + \gamma E_1 + \delta E_2 + M_3$, $E.E_1 = E.E_2 = E_1.E_2 = 1$, $\beta \in \{2,3\}$, $32 \leq H^2 \leq 52$ or $H^2 = 28$.
(III) $H \equiv \beta E + \gamma E_1 + M_2$, $E.E_1 = 2$, $\beta \in \{2,3\}$, $32 \leq H^2 \leq 62$ or $H^2 = 28$

(where the limitations on β are obtained using the same B's as above), in addition to:

(D) $H \equiv 2H_1$ for some $H_1 > 0$, $H_1^2 \geq 8$,
(S) L_1 is of small type and $H^2 \geq 32$ or $H^2 = 28$.

Definition 6.1. *We call such decompositions as in* (I), (II) *and* (III)*, obtained by the inductive process and removing conventions above, a* **ladder decomposition** *of* H*.*

Note that $M_n, n = 2,3$, satisfies: $M_n \geq 0$, $M_n^2 \geq 0$ and M_n is of small type. Moreover, when $M_n > 0$, we will replace M_n with $M_n + K_S$ that has the same properties, to avoid to study the two different numerical equivalence classes for H. Also note that $\beta \geq \alpha \geq 2$ and $\beta \geq \alpha + 2$ in (I-A).

We will treat all these cases separately in the next sections.

A useful fact will be the following

Lemma 6.2. *If $E.E_1 \leq 2$, then $E + E_1$ is nef.*

Proof. Let Γ be a nodal curve such that $\Gamma.(E+E_1) < 0$. Since E is nef, we must have $\Gamma.E_1 < 0$. By Lemma 4.11 we can then write $E_1 = A + k\Gamma$, for $A > 0$ primitive with $A^2 = 0$, $k = -\Gamma.E_1 \geq 1$ and $\Gamma.A = k$. Since $A.L_1 \geq \phi(L_1) = E_1.L_1$, we get $k\Gamma.L_1 = (E_1 - A).L_1 \leq 0$, whence $\Gamma.E > 0$, because H is ample. This yields $k \geq \Gamma.E + 1 \geq 2$. Hence $E.E_1 = E.A + k\Gamma.E \geq 2\Gamma.E$, and we get $k = 2$, $\Gamma.E = 1$ and $E.A = 0$. Hence $A \equiv E$ by Lemma 4.2, contradicting the fact that $\Gamma.A = 2$. □

From [CD, Prop.3.1.6, 3.1.4 and Thm.4.4.1] and the above Lemma we now know that $E + E_1$ is base-point free when $E.E_1 = 2$, and that $E + E_1$ is base-component free when $E.E_1 = 1$, unless $E_1 \sim E + R$, for a nodal curve R such that $E.R = 1$. But since we are free to choose between E_1 and $E_1 + K_S$ (they both calculate $\phi(L_1)$), **we adopt the convention of always choosing E_1 such that $E + E_1$ is base-component free**. We therefore have

Lemma 6.3. *If $E.E_1 = 2$, then $E + E_1$ is base-point free.*

If $E.E_1 = 1$, then $E + E_1$ is base-component free. Furthermore if there exists $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.E_1 < 0$, then Δ is a nodal curve and $E_1 \sim E + \Delta + K_S$.

Moreover in both cases we have $H^1(E_1) = H^1(E_1 + K_S) = 0$.

Proof. We need to prove the last two assertions. Suppose there exists a $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.E_1 < 0$. By Lemma 4.11 we can write $E_1 = A + k\Delta$, for $A > 0$ primitive with $A^2 = 0$ and $k = -\Delta.E_1 = \Delta.A \geq 1$. Now $0 \leq (E + E_1).\Delta = E.\Delta - k$ gives $E.\Delta \geq k$. From $2 \geq E.E_1 = E.A + kE.\Delta \geq k^2$ we get $k = 1$, whence E_1 is quasi-nef and primitive, so that the desired vanishing follows by Theorem 4.13. Now if $E.E_1 = 1$ we get $E.A = 0$, whence $A \equiv E$ by Lemma 4.2 and then $E_1 \equiv E + \Delta$. Since E_1 is not nef, by [CD, Prop.3.1.4, Prop.3.6.1 and Cor.3.1.4] there is a nodal curve R such that $E_1 \sim E + R + K_S$, whence $\Delta = R$. □

Another useful nefness lemma is the following.

Lemma 6.4. *Let $H \sim \beta E + \gamma E_1 + M_2$ be of type (I) or (III), with $M_2 > 0$ and $M_2^2 \leq 4$. Let $i = 2$ and $M_2 \sim E_2$ or $i = 2, 3$ and $M_2 \sim E_2 + E_3$ be genus 1 decompositions of M_2 (note that, by construction, $E.E_j \geq 1$ for $j = 1, 2$).*

Assume that E_i is quasi-nef. Then:

(a) *$|2E + E_1 + E_i|$ is base-point free.*
(b) *$|E + E_1 + E_i|$ is base-point free if $\beta = 2$ or if $E.E_1 = 1$ and $E_1.E_i \neq E.E_i - 1$.*
(c) *Assume $\gamma = 2$ and $E.E_1 = E_1.E_i = 1$. Then $E + E_i$ is nef if either $E.E_i \geq 2$ or if $M_2^2 \geq 2$ and $E_1.M_2 \geq 4$.*
(d) *Assume $\gamma = 2$, $M_2^2 = 2$, $E.E_1 = E_1.E_2 = E_1.E_3 = 1$ and that both E_2 and E_3 are quasi-nef. Then either $E + E_2$ or $E + E_3$ is nef.*
(e) *If $E.E_1 = E.E_i = 1$ and $E_1.E_i \neq 1$ then $E_1 + E_i$ is nef.*

Proof. Assume there is a nodal curve R with $R.(E + E_1 + E_i) < 0$. Then Lemma 6.2 and the quasi-nefness of E_i yield $R.(E + E_1) = 0$ and $R.E_i = -1$. Moreover $R.E_1 \leq 0$ by the nefness of E.

If $R.E = 0$ or if $\beta = 2$, then $R.H = R.(\beta E + \gamma E_1 + M_2) \leq R.M_2$ implies $M_2^2 \geq 2$ and $R.E_j \geq 2$ for $j \in \{2, 3\} - \{i\}$. By Lemma 4.11 we can write $E_i \sim A + R$ with $A > 0$ primitive, $A^2 = 0$ and $A.R = 1$. From $2 \geq E_j.E_i = A.E_j + R.E_j \geq A.E_j + 2$, we get $A \equiv E_j$ by Lemma 4.2 and the contradiction $1 = R.A = R.E_j = 2$.

Therefore $R.E > 0$ and $\beta \geq 3$, so that $R.E_1 < 0$. By Lemma 4.11 we can write $E_1 \sim A + kR$ with $A > 0$ primitive, $A^2 = 0$ and $k := -R.E_1 = A.R \geq 1$. Now $2 \geq E.E_1 = A.E + kE.R \geq A.E + k$ gives $k = 1$ by Lemma 4.2. Hence $2E + E_1 + E_i$ is nef, whence base-point free, as $\phi(2E + E_1 + E_i) \geq 2$, and (a) is proved. Moreover, if $E.E_1 = 1$, then $E_1 \equiv E + R$ by Lemma 6.3, whence $E_1.E_i = E.E_i - 1$, and (b) is proved, again since $\phi(E + E_1 + E_i) \geq 2$.

To prove (c), assume $\gamma = 2$ and $E.E_1 = E_1.E_i = 1$ and that there is a nodal curve R with $R.(E + E_i) < 0$. Then $R.E = 0$ and $R.E_i = -1$ by quasi-nefness, and moreover $R.E_1 > 0$ by (a). Therefore $E_1.E_i = 1$ yields $E_i \equiv E_1 + R$ with $R.E_1 = 1$, so that $E.E_i = E.E_1 = 1$. Moreover, if $M_2^2 \geq 2$ and $j \in \{2, 3\} - \{i\}$, then $R.H = R.(\beta E + 2E_1 + E_i + E_j) = 1 + R.E_j$, so that $R.E_j \geq 0$. It follows that $E_j.E_1 = E_j.(E_i - R) \leq E_j.E_i \leq 2$, whence $E_1.M_2 = E_1.(E_i + E_j) \leq 3$ and (c) is proved.

Moreover, by what we have just seen, under the assumptions in (d), if neither $E + E_2$ nor $E + E_3$ is nef, then $E_2 \equiv E_1 + R_2$ and $E_3 \equiv E_1 + R_3$ with R_2 and R_3 nodal curves such that $R_2.E_1 = R_3.E_1 = 1$. Then $R_2.R_3 = (E_2 - E_1).(E_3 - E_1) = -1$, a contradiction. This proves (d).

Finally, to prove (e), assume that there is a nodal curve R such that $R.(E_1 + E_i) < 0$. By Lemma 4.11 we can write $E_1 + E_i \sim B + kR$ with $B > 0$, $B^2 = 2E_1.E_i > 0$ and $k := -R.(E_1 + E_i) \geq 1$. By (a) we have $E.R \geq 1$. From $2 = E.(E_1 + E_i) = E.B + kE.R \geq 1 + k \geq 2$, we get $k = E.R = 1$, so that $R.(E_1 + E_i) = -1$. Now if $E_1.R \leq -1$ we get, by Lemma 6.3, that $E_1 \equiv E + R$, $E_1.R = -1$ and $E_i.R = 0$, giving the contradiction $E_1.E_i = 1$. If $E_1.R \geq 0$ the quasi-nefness of E_i implies

that $E_1.R = 0$ and $E_i.R = -1$, whence $E_i \equiv E + R$ by Lemma 4.11, giving again the contradiction $E_1.E_i = 1$. □

The general strategy to prove the nonextendability of S in the remaining cases (I), (II), (III), (D) and (S), will be as follows.

We will first use the ladder decomposition and Propositions 5.2-5.5 to reduce to particular genus one decompositions of M_2 or M_3 where we know all the intersections involved.

Then we will find a big and nef line bundle D_0 on S such that $\phi(D_0) \geq 2$ and $H - D_0$ is base-component free with $(H - D_0)^2 > 0$. In particular, this implies that $H^1(H - D_0) = H^1(D_0 - H) = 0$. In many cases this D_0 will satisfy the numerical conditions in Lemma 5.6, so that S will be nonextendable.

In the remaining cases we will apply Proposition 5.1 in the following way.

We will let $D \in |D_0|$ be a general smooth curve. (This will be used repeatedly without further mentioning.)

The surjectivity of the Gaussian map Φ_{H_D,ω_D} will be handled by means of Theorem 5.3, to which we will refer. Moreover in all of the cases where we will apply Theorem 5.3 (with the exception of (e)) we will have that $h^0(\mathcal{O}_D(2D_0 - H)) \leq 1$ if $D_0^2 \geq 6$ and $h^0(\mathcal{O}_D(2D_0 - H)) = 0$ if $D_0^2 = 4$. Therefore the hypothesis (iii) of Proposition 5.1 will always be satisfied and we will skip its verification.

To study the surjectivity of the multiplication map μ_{V_D,ω_D} we will use several tools, outlined below. In several cases we will find an effective decomposition $D \sim D_1 + D_2$ and use Lemma 5.7. We remark that **except possibly for the one case in** (17) **below where** D_1 **is primitive of canonical type, both** D_1 **and** D_2 **will always be smooth curves**. The reason for this is that we will always have that $|D_1|$ and $|D_2|$ are base-component free and not multiple of elliptic pencils, whence their general members will be smooth and irreducible [CD, Prop.3.1.4 and Thm.4.10.2]. In most cases this will again be used without further mentioning.

Furthermore the spaces V_D, V_{D_1} and V_{D_2} will always be base-point free. This is immediately clear for V_D, as $|D_0|$ is base-point free. As for V_{D_1} and V_{D_2}, one only has to make sure that, in the cases where $|H - D_0|$ has base points (that is, $\phi(H - D_0) = 1$), in which case it has precisely two distinct base points [CD, Prop.3.1.4 and Thm.4.4.1], they do not intersect the possible base points of $|D_1|$ and $|D_2|$. This will always be satisfied, and again, we will not repeatedly mention this.

Here are the criteria we will use to verify that the desired multiplication maps are surjective:

μ_{V_D,ω_D} is surjective in any of the following cases:

(14) $H^1(H - 2D_0) = 0$ and either $|D_0|$ or $|H - D_0|$ is birational (see Rmk. 6.5 below).

(15) $H^1(H - 2D_0) = 0$ and $|H - D_0|$ is a pencil.

If V_{D_1} is base-point free, then $\mu_{V_{D_1},\omega_{D_1}}$ is surjective in any of the following cases:

(16) $H^1(H - D_0 - D_1) = 0$, D_1 is smooth and $(H - D_0).D_1 \geq D_1^2 + 3$;

(17) $H^1(H - D_0 - D_1) = 0$ and D_1 is nef and isotropic.

If D_2 is smooth and V_{D_2} is base-point free, then $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective if

(18) $h^0(H - D_0 - D_2) + h^0(\mathcal{O}_{D_2}(H - D_0 - D_1)) \leq \frac{1}{2}(H - D_0)^2 - 1$ (see Rmk. 6.6).

To see (14)-(15) note that if $H^1(H - 2D_0) = 0$, we have that $V_D = H^0((H - D_0)_{|D})$, whence (15) is just the base-point free pencil trick, while (14) follows from the base-point free pencil trick and [AS, Thm.1.6] since $(H - D_0)_{|D}$ is base-point free and is either a pencil or birational by any of the hypotheses. The same proves (16). As for (17) the hypotheses imply $V_{D_1} = H^0((H - D_0)_{|D_1})$ and $\omega_{D_1} \cong \mathcal{O}_{D_1}$, as D_1 is either a smooth elliptic curve or indecomposable of canonical type [CD, III, §1], and the surjectivity is immediate.

Finally for (18) we use the H^0-lemma [Gr, Thm.4.e.1], which states that $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective if

$$\begin{aligned}\dim V_{D_2}-2 &= h^0(H-D_0)-h^0(H-D_0-D_2)-2\geq \\ &\geq h^1(\omega_{D_2}(D_1-(H-D_0))=h^0(\mathcal{O}_{D_2}(H-D_0-D_1)).\end{aligned}$$

Using Riemann-Roch on S, this is equivalent to (18).

Remark 6.5. A complete linear system $|B|$ is birational if it defines a birational map. By [CD, Prop.3.1.4, Lemma4.6.2, Thm.4.6.3, Prop.4.7.1 and Thm.4.7.1] a nef divisor B with $B^2\geq 8$ defines a birational morphism if $\phi(B)\geq 2$ and B is not 2-divisible in $\operatorname{Pic} S$ when $B^2=8$.

Remark 6.6. The inequality in (18) will be verified by giving an upper bound on $h^0(H-D_0-D_2)$ and using Riemann-Roch and Clifford's theorem on D_2 to bound $h^0(\mathcal{O}_{D_2}(H-D_0-D_1))$. We will often not mention this.

7. Case (D)

We have $H\equiv 2H_1$ whence H_1 is ample with $H_1^2\geq 8$ and $\phi(H)=2\phi(H_1)\geq 3$ gives $\phi(H_1)\geq 2$.

7.1. $H\sim 2H_1+K_S$**.** We will prove that $S\subset\mathbb{P}^r$ is nonextendable. We set $D_0=H_1$ and check that the hypotheses of Proposition 5.1 are satisfied by D_0. Of course D_0 is ample and $\phi(D_0)\geq 2$. The Gaussian map Φ_{H_D,ω_D} is surjective by Theorem 5.3(c) since $H^0(2D_0-H)=H^0(K_S)=0$. Moreover also $H^1(H-2D_0)=0$, whence and $(H-D_0)_{|D}=\omega_D$ so that the multiplication map μ_{V_D,ω_D} is just μ_{ω_D,ω_D} which is surjective since D is not hyperelliptic.

7.2. $H\sim 2H_1$**.**

7.2.1. $\phi(H_1)\geq 3$. In the course of the proof of Corollary 2.4 we have seen that, given an extendable Enriques surface $S\subset\mathbb{P}^r$, we can reembed it in such a way that it is linearly normal and still extendable. Therefore we can assume that $S\subset\mathbb{P}H^0(2H_1)$. By [CD, Cor.2, page 283] it follows that H_1 is very ample, whence $S\subset\mathbb{P}H^0(2H_1)$ is 2-Veronese of $S_1=\varphi_{H_1}(S)\subset\mathbb{P}H^0(H_1)$ and therefore $S\subset\mathbb{P}^r$ is nonextendable by [GLM, Thm.1.2].

7.2.2. $\phi(H_1)=2$ *and* $H_1^2=8$. Since $E.H_1=2$ we have $(H_1-2E)^2=0$ and we can write $H_1=2E+F$ with $F>0, F^2=0, E.F=2$. According to whether F is primitive or not, we get cases (a1) and (a2) in the proof of Proposition 15.1.

7.2.3. $\phi(H_1)=2$ *and* $H_1^2=10$. Since $E.H_1=2$ we have $(H_1-2E)^2=2$ and we can write $H_1=2E+E_1+E_2$ with $E_i>0, E_i^2=0, E.E_i=E_1.E_2=1$. Then $H\sim 4E+2E_1+2E_2$ with $\phi(H)=4$. Note that $\alpha=2$ and $E_1.L_1=E_2.L_1=\phi(L_1)$.

First we show that either E_1 or E_2 is nef. If not, by Lemma 6.3, we have $E_1\equiv E+\Gamma_1$, $E_2\equiv E+\Gamma_2$ for two nodal curves Γ_i with $\Gamma_i.E=1$. But then $\Gamma_1.\Gamma_2=(E_1-E).(E_2-E)=-1$, a contradiction. Therefore, replacing E_1 with E_2 if necessary, we can assume that E_1 is nef. Moreover, possibly after substituting E_2 with E_2+K_S, we can assume that either E_2 is nef or there exists a nodal curve Γ such that $E_2\sim E+\Gamma+K_S$. In particular $E+E_2$ is base-component free.

We set $D_0=E+2E_1+E_2$, which is nef with $\phi(D_0)=2$, $D_0^2=10$ and $H-D_0\sim 3E+E_2$ is base-component free. Moreover $2D_0-H\sim -2E+2E_1$ and $h^0(2D_0-H)=0$ by the nefness of E_1, whence Φ_{H_D,ω_D} is surjective by Theorem 5.3(c).

Now to see the surjectivity of μ_{V_D,ω_D} note that since $h^1(-2E_1)=h^1(K_S)=0$ the two restriction maps $H^0(E+E_2)\to H^0(\mathcal{O}_D(E+E_2))$ and $H^0(E+2E_1+E_2+K_S)\to H^0(\omega_D)$ are surjective and $|\mathcal{O}_D(E+E_2)|$ is a base-point free pencil.

Consider the commutative diagram

$$\begin{tikzcd}
H^0(2E) \otimes H^0(E+E_2) \otimes H^0(E+2E_1+E_2+K_S) \arrow[r, "r_D"] \arrow[d, "\mu_{2E,E+E_2}"] & W_D \otimes H^0(\mathcal{O}_D(E+E_2)) \otimes H^0(\omega_D) \arrow[d, "\mathrm{Id} \otimes \mu_{\mathcal{O}_D(E+E_2),\omega_D}"] \\
H^0(H-D_0) \otimes H^0(D_0+K_S) \arrow[d, "p_D"] & W_D \otimes H^0(\omega_D(E+E_2)) \arrow[d, "\mu_{W_D,\omega_D(E+E_2)}"] \\
V_D \otimes H^0(\omega_D) \arrow[r, "\mu_{V_D,\omega_D}"] & H^0(\mathcal{O}_D(H+K_S)),
\end{tikzcd}$$

where p_D and r_D are the natural restriction maps and $W_D := \mathrm{Im}\{H^0(2E) \to H^0(\mathcal{O}_D(2E))\}$. The map r_D is surjective and the map $\mu_{\mathcal{O}_D(E+E_2),\omega_D}$ is surjective by the base-point free pencil trick. Now to prove that μ_{V_D,ω_D} is surjective it suffices to show that $\mu_{W_D,\omega_D(E+E_2)}$ is surjective.

Since $\dim W_D = 2$ and W_D is base-point free, the surjectivity of $\mu_{W_D,\omega_D(E+E_2)}$ follows by the H^0-lemma [Gr, Thm.4.e.1] as soon as we prove that $h^1(\omega_D(E_2-E)) = 0$. Now $h^1(\omega_D(E_2-E)) = h^0(\mathcal{O}_D(E-E_2))$ and, since $\deg \mathcal{O}_D(E-E_2) = 0$, the required vanishing follows unless $\mathcal{O}_D(E-E_2)$ is trivial. But if the latter holds, then we have a short exact sequence

$$0 \longrightarrow -2E_1 - 2E_2 \longrightarrow E - E_2 \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

If E_2 is nef then $h^0(E-E_2) = h^1(-2E_1-2E_2) = 0$ and we get the contradiction $h^0(\mathcal{O}_D) = 0$.

If E_2 is not nef then $E_2 \sim E + \Gamma + K_S$, and $\Gamma.D = 0$ whence $h^1(\omega_D(E_2-E)) = h^1(\mathcal{O}_D(D_0+\Gamma)) = h^1(\mathcal{O}_D(D_0)) = 0$.

Therefore in both cases μ_{V_D,ω_D} is surjective and S is nonextendable by Proposition 5.1.

7.2.4. *$\phi(H_1) = 2$ and $H_1^2 \geq 12$.* We set $D_0 = H_1$ and check that the hypotheses of Proposition 5.1 are satisfied. Of course D_0 is ample and $\phi(D_0) = 2$. The Gaussian map Φ_{H_D,ω_D} is surjective by Theorem 5.3(d) since $h^0(2D_0 - H) = h^0(\mathcal{O}_S) = 1$. Moreover also $H^1(H-2D_0) = 0$, whence μ_{V_D,ω_D} is surjective by (14). Hence $S \subset \mathbb{P}^r$ is nonextendable.

8. Case (I) with $M_2 = 0$

We have $H \equiv \beta E + \gamma E_1$ with $E.E_1 = 1, \beta \geq 2, \gamma \in \{2,3\}$ and $H^2 \geq 32$ or $H^2 = 28$.

Now $\gamma = E.H = \phi(H) \geq 3$, whence $\gamma = 3$ and we will deal with $H \equiv \beta E + 3E_1, E.E_1 = 1, \beta \geq 6$.

We set $D_0 = \lfloor \frac{\beta}{2} \rfloor E + 2E_1$ and check the conditions of Proposition 5.1. We have that D_0 is nef by Lemma 6.2, $\phi(D_0) = 2$ and $D_0^2 = 4\lfloor \frac{\beta}{2} \rfloor \geq 12$. Now $H - D_0 \equiv \lfloor \frac{\beta+1}{2} \rfloor E + E_1$ and $2D_0 - H \equiv E_1$(respectively $E_1 - E$) if β is even (respectively if β is odd). Therefore, by Lemma 6.3, we have that $h^0(2D_0 - H) \leq 1$ and the Gaussian map Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d). To prove the surjectivity of the multiplication map μ_{V_D,ω_D} we apply Lemma 5.8. We have $D_0 + K_S - 2E = (\lfloor \frac{\beta}{2} \rfloor - 2)E + 2E_1 + K_S$ whence $H^1(D_0 + K_S - 2E) = 0$ by Lemma 6.3. Also $H^2(D_0 + K_S - 4E) = 0$ since $E.(D_0 + K_S - 4E) = 2$. Now $H - D_0 - 2E \equiv \lfloor \frac{\beta-3}{2} \rfloor E + E_1$ is base-component free by Lemma 6.3 and $H - 2D_0 - 2E \equiv (\beta - 2\lfloor \frac{\beta}{2} \rfloor - 2)E - E_1$, whence $h^0(H - 2D_0 - 2E) = 0$ by the nefness of E. Also $H - 2D_0 - 4E \equiv -\varepsilon E - E_1, \varepsilon = 3, 4$, whence $H^1(H - 2D_0 - 4E) = 0$ and the exact sequence

$$0 \longrightarrow H - 2D_0 - 4E \longrightarrow H - D_0 - 4E \longrightarrow \mathcal{O}_D(H - D_0 - 4E) \longrightarrow 0$$

shows that $h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq h^0(H - D_0 - 4E)$. Since $H - D_0 - 4E \equiv \lfloor \frac{\beta-7}{2} \rfloor E + E_1$ we have that $h^0(H - D_0 - 4E) = \lfloor \frac{\beta-5}{2} \rfloor$ if $\beta \geq 7$, while if $\beta = 6$, we have $H - D_0 - 4E \equiv -E + E_1$ and replacing D_0 with $D_0 + K_S$ if necessary, we can assume that $h^0(H - D_0 - 4E) = 0$ by Lemma 6.3. In both cases we get that (11) is satisfied.

9. Case (I) with $\gamma = 3$ and $M_2 > 0$

We first note that, since $\gamma = 3$, we must have $\beta \geq 3$. Indeed, if $\beta = 2$ we have $L_2 \sim E_1 + M_2$ and $E.L_2 = 1 + E.M_2 = \phi(H) - 2 \leq E_1.H - 2 = E_1.M_2 = E_1.L_2$, contradicting the removing conventions of Section 6, page 19 (because then $(L_2 - E)^2 \geq (L_2 - E_1)^2 \geq 0$, therefore we could find E in a genus 1 decomposition of L_2, but then $\beta \geq 3$). Then we have, in this section,

$$H \sim \beta E + 3E_1 + M_2, \quad E.E_1 = 1, \quad \beta \geq 3, \ M_2 > 0, \quad M_2^2 \geq 0 \ \text{ and } \ H^2 \geq 32 \ \text{ or } \ H^2 = 28. \tag{19}$$

We will use the following:

Lemma 9.1. *Let $N = E + E_1$. Then either $|H - 2N|$ is base-point free and $h^1(H - 3N) = 0$, or S is nonextendable.*

Proof. We have $H - 3N \sim (\beta - 3)E + M_2 > 0$ and $(H - 3N)^2 \geq 0$ with equality only if $H - 3N = M_2$. Since M_2 is of small type it follows by Theorem 4.13 that $h^1(H - 3N) = 0$ if and only if $H - 3N$ is quasi-nef. Moreover, as $(H - 2N)^2 = 2(\beta - 2)(E.M_2 + 1) + 2E_1.M_2 + M_2^2 \geq 6$ and $\phi(H - 2N) \geq 2$, we have that $H - 2N$ is base-point free if and only if it is nef. Now if $H - 2N$ is not nef, there is a nodal curve Γ with $\Gamma.(H - 2N) < 0$. Then $\Gamma.N > 0$, since H is ample, whence $\Gamma.(H - 3N) \leq -2$.

Therefore, to prove the lemma, it suffices to show that S is nonextendable if $H - 3N$ is not quasi-nef.

Let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.(H - 3N) \leq -2$. We have $\Delta.N > 0$ since H is ample. Also note that $\Delta.E_1 \geq 0$, for if not, we would have $\Delta.E \geq 2$, whence the contradiction $(E + \Delta)^2 \geq 2$ and $E_1.(E + \Delta) \leq 0$. Hence $M_2.\Delta \leq -2$ and by Lemma 4.11 we can write $M_2 \sim A + k\Delta$, with $A > 0$ primitive, $A^2 = M_2^2$ and $k := -\Delta.M_2 = \Delta.A \geq 2$. Now if $E.\Delta > 0$ we find $E.M_2 \geq k$ and if equality holds, then $E.A = 0$ and $E.\Delta = 1$, whence $E \equiv A$ by Lemma 4.2, a contradiction. We get the same contradiction if $E_1.\Delta > 0$. Therefore

$$E.M_2 \geq -\Delta.M_2 + 1 \geq 3 \text{ if } E.\Delta > 0 \text{ and } E_1.M_2 \geq -\Delta.M_2 + 1 \geq 3 \text{ if } E_1.\Delta > 0. \tag{20}$$

We first consider the case $E.\Delta > 0$.

Note that we cannot have that $\beta = 3$, for otherwise H is of type (I-B) in Section 6 and $L_2 \sim (3 - \alpha)E + E_1 + M_2$ is of small type, whence $E_1.M_2 \leq 5$ by Lemma 4.8, so that $E_1.(H - 2E) = E_1.(E + 3E_1 + M_2) \leq 6$. Since $\phi(H) = E.H = 3 + E.M_2 \geq 6$ by (20), we get $\alpha = 2$ and $E_1.H = 3 + E_1.M_2 \geq 6$, so that $E_1.M_2 \geq 3$. Hence $L_2 \sim E + E_1 + M_2$ and $L_2^2 \geq 14$, a contradiction.

Therefore $\beta \geq 4$, whence $\Delta.M_2 \leq -2 - (\beta - 3)\Delta.E \leq -3$, so that $E.M_2 \geq 4$ by (20) and $\phi(H) \geq 7$, whence $H^2 \geq 54$ by [KL2, Prop.1]. Now one easily verifies that $B := 2E + E_1 + \Delta$ satisfies the conditions in Proposition 5.2, so that S is nonextendable.

We finally consider the case $\Delta.E = 0$, where $E_1.\Delta > 0$, so that $E_1.M_2 \geq 3$ by (20).

Now $L_2 \sim (\beta - \alpha)E + E_1 + M_2$ if H is of type (I-B) in Section 6 and $L_3 \sim (\beta - \alpha - 2)E + E_1 + M_2$ if H is of type (I-A) in Section 6. We claim that the removing conventions of Section 6, page 19 now imply that $E_1.M_2 \leq E.M_2 + 1$ and, if $\beta = 3$, that $E_1.M_2 \leq E.M_2$. In fact if the latter inequalities do not hold we have that $E.L_2 \leq E_1.L_2$, $E.L_3 < E_1.L_3$ and $(E_1 + M_2 - E)^2 \geq 0$, contradicting the fact that L_2 and L_3 are of small type. To summarize, we must have

$$E.M_2 \geq 2, \text{ and furthermore } E.M_2 \geq 3 \text{ if } \beta = 3. \tag{21}$$

This yields $H^2 = 6\beta + 2\beta E.M_2 + 6E_1.M_2 + M_2^2 \geq 54$ in any case. Moreover, using (21), one easily verifies that $B := E + 2E_1 + \Delta$ satisfies the conditions in Proposition 5.2, so that S is nonextendable. $\square$

The main result of this section is the following:

Proposition 9.2. *If H is of type* (I) *with $\gamma = 3$ and $M_2 > 0$ then S is nonextendable.*

Proof. Set $D_0 = 2N = 2(E + E_1)$, which is nef by Lemma 6.2 with $D_0^2 = 8$ and $\phi(D_0) = 2$. By Lemma 9.1 we can assume that $H - D_0$ is base-point free as well.

By assumption we have $H.D_0 = 2(\beta + 3 + (E + E_1).M_2) \geq 16$ with equality only if $\beta = 3$ and $E.M_2 = E_1.M_2 = 1$. But in the latter case, since M_2 does not contain E in its arithmetic genus 1 decompositions, we have, by Lemma 4.4, that $M_2^2 = 0$ and $H^2 = 30$, a contradiction. Hence $(2D_0 - H).D_0 < 0$ and consequently $h^0(2D_0 - H) = 0$, so that Φ_{H_D,ω_D} is surjective by Theorem 5.3(c).

Let $D_1, D_2 \in |N|$ be two general members. By Lemma 9.1 we can assume $h^1(H - D_0 - D_i) = 0$. Hence $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (16) since $(H - D_0).D_1 = \beta - 1 + (E + E_1).M_2 \geq 5$ by our assumptions, and the map $\mu_{V_{D_2},\omega_{D_2}(D_1)} = \mu_{\mathcal{O}_{D_2}(H-D_0),\omega_{D_2}(D_1)}$ is surjective by [Gr, Cor.4.e.4] since $\deg \omega_{D_2}(D_1) = 2g(D_2)$ and $\deg \mathcal{O}_{D_2}(H - D_0) \geq 2g(D_2) + 1$.

By Lemma 5.7, μ_{V_D,ω_D} is surjective and by Proposition 5.1, S is nonextendable. □

10. Case (I) with $\gamma = 2$ and $M_2 > 0$

We have

$$H \sim \beta E + 2E_1 + M_2, \quad E.E_1 = 1, \quad M_2 > 0, \quad M_2^2 \geq 0, \quad H^2 \geq 32 \text{ or } H^2 = 28$$

and, as above, either L_2 or L_3 is of small type, whence so is M_2.

Recall that $E_1.M_2 \leq E_1.M_2 + \beta - \alpha = E_1.L_1 = \phi(L_1) \leq \phi(H) = 2 + E.M_2 \leq E_1.H = \beta + E_1.M_2$. Moreover, since by construction M_2 neither contains E nor E_1 in its arithmetic genus 1 decompositions, we have, by Lemma 4.4, that $(M_2 - E)^2 < 0$ and $(M_2 - E_1)^2 < 0$. Hence

$$\frac{1}{2}M_2^2 + 1 \leq \quad E.M_2 \quad \leq E_1.M_2 + \beta - 2, \quad \text{and} \tag{22}$$

$$\frac{1}{2}M_2^2 + 1 \leq \quad E_1.M_2 \quad \leq E.M_2 + 2 - \beta + \alpha \leq E.M_2 + 2. \tag{23}$$

In this section we will first prove, in Lemma 10.1, nonextendability up to some explicit decompositions of H and we will then prove, in Proposition 10.3, nonextendability if $\beta \geq 5$.

Lemma 10.1. *Let H be of type* (I) *with $\gamma = 2$ and $M_2^2 \geq 2$. Then S is nonextendable unless, possibly, we are in one of the following cases (where all the E_i's are effective and isotropic):*

(a) $M_2^2 = 2$, $M_2 \sim E_2 + E_3$, $E_2.E_3 = 1$, *and either*
  (a-i) $\beta = 2$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 1, 2), (1, 2, 2, 1), (1, 1, 2, 2), (2, 2, 2, 2), (1, 2, 2, 2)$*; or*
  (a-ii) $\beta = 3$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (2, 2, 2, 2), (2, 2, 1, 2)$*; or*
  (a-iii) $\beta \geq 3$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2)$.
(b) $M_2^2 = 4$, $M_2 \sim E_2 + E_3$, $E_2.E_3 = 2$, *and either*
  (b-i) $\beta = 2$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 1, 2), (1, 2, 2, 1), (1, 2, 2, 2), (1, 2, 1, 3)$*; or*
  (b-ii) $\beta = 3$, $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 2, 1), (1, 2, 1, 3)$.
(c) $M_2^2 = 6$, $M_2 \sim E_2 + E_3 + E_4$, $E_2.E_3 = E_2.E_4 = E_3.E_4 = 1$, *and*
  $\beta = 2$, $(E.E_2, E.E_3, E.E_4, E_1.E_2, E_1.E_3, E_1.E_4) = (1, 1, 2, 1, 1, 2)$.

Proof. We write $M_2 \sim E_2 + \ldots + E_{k+1}$ as in Lemma 4.8 with $k = 2$ or 3. Moreover we can assume that $1 \leq E.E_2 \leq \ldots \leq E.E_{k+1}$, whence that $E.M_2 \geq kE.E_2$.

We first consider the case $\beta \geq 4$.

We note that $(M_2-2E_2)^2 = -2$ if $M_2^2 = 2$ or 6, $(M_2-2E_2)^2 = -4$ if $M_2^2 = 4$ and $(M_2-2E_2)^2 \geq -6$ if $M_2^2 = 10$. In the latter case $E.M_2 \geq 6$ by (22), whence $E.(M_2 - 2E_2) \geq 2$. Using this and setting $B := E + E_1 + E_2$ one easily verifies that $(H - 2B)^2 = 2(\beta - 2)E.(M_2 - 2E_2) + (M_2 - 2E_2)^2 \geq 0$ and $E.(H - 2B) > 0$ (whence $H - 2B \geq 0$ by Riemann-Roch), except for the cases

$$M_2^2 = 2, 4 \text{ and } E.E_2 = E.E_3. \tag{24}$$

Moreover, except for these cases, using (22) and (23), one easily verifies that $H^2 \geq 54$, except for the case $\beta = 4$, $M_2^2 = 2$ and $(E.M_2, E_1.M_2) = (3, 2)$, where $H^2 = 50$. In this case $(3B - H).H = 4 < \phi(H) = 5$, so that, if $3B - H > 0$ it must be a nodal cycle. Therefore either $h^0(3B - H) = 0$ or $h^0(3B + K_S - H) = 0$, so in any case B satisfies the conditions in Propositions 5.2 or 5.4 and S is nonextendable.

In the remaining cases (24) we can without loss of generality assume $1 \leq E_1.E_2 \leq E_1.E_3$ and we set $B := E + E_2$. Then $(H - 2B)^2 = 4(\beta - 2) + 4E_1.(E_3 - E_2) + (E_3 - E_2)^2 \geq 4$ and $(H - 2B).E = 2$. Using (22) and (23), one gets $H^2 \geq 64$ if $M_2^2 = 4$, $H^2 \geq 74$ if $M_2^2 = 2$ and $E.E_2 = E.E_3 = 3$, and $B.H \geq 17$ if $M_2^2 = 2$ and $E.E_2 = E.E_3 = 2$. Moreover, in the latter case, we have that again $H^2 \geq 64$ unless $\beta = 4$ and $E_1.M_2 = 2, 3$, which gives $E_1.E_2 = 1$ and B is nef by Lemma 6.4(c) since $E_2.H = 11 < 2\phi(H) = 12$, whence E_2 is quasi-nef by Lemma 4.14. Therefore B satisfies the conditions in Propositions 5.2 or 5.5 and S is nonextendable unless $M_2^2 = 2$ and $E.E_2 = E.E_3 = 1$. In the latter case, by (23) we have $2 \leq E_1.M_2 \leq 4$ with $E_1.M_2 = 4$ if and only if $\alpha = \beta$. In this last case $L_1 \sim 2E_1 + M_2$, whence $\phi(L_1) = E_1.M_2 = 4$ and we get that $4 \leq E_i.L_1 = 2E_1.E_i + 1$ for $i = 2, 3$, so that $E_1.E_2 = E_1.E_3 = 2$. Therefore we get the cases in (a-iii) with $\beta \geq 4$.

We next treat the cases $\beta \leq 3$. Then we must be in case (I-B) of Section 6, whence L_2 is of small type and either $L_2 \sim M_2$ or $L_2 \sim E + M_2$.

Suppose first that $L_2 \sim E + M_2$.

Then $\beta \geq \alpha + 1 \geq 3$, whence $\beta = 3$, $\alpha = 2$ and, since L_2 is of small type, by (22), we can only have $(M_2^2, E.M_2) = (2, 2)$, $(2, 4)$ or $(4, 3)$.

If $(M_2^2, E.M_2) = (2, 2)$, then $E.E_2 = E.E_3 = 1$ and by (23) we have $2 \leq E_1.M_2 \leq 3$, yielding the first two cases in (a-iii).

If $(M_2^2, E.M_2) = (2, 4)$, then $L_2^2 = 10$ and $\phi(L_2) = 3$. As $E.E_i + 1 = L_2.E_i \geq \phi(L_2) = 3$ for $i = 2, 3$, we must have $E.E_2 = E.E_3 = 2$. Now $L_1 \sim E + 2E_1 + M_2$ and $(1 + E_1.M_2)^2 = \phi(L_1)^2 \leq L_1^2 = 14 + 4E_1.M_2$ and (22) yield $E_1.M_2 = 3$ or 4. Therefore, by Lemma 4.3 and symmetry, we get the two cases in (a-ii).

If $(M_2^2, E.M_2) = (4, 3)$, then $E_1.M_2 = 3$ or 4 by (23). Since $L_2^2 = 10$ and $\phi(L_2) = E.L_2 = 3$, there is by [CD, Cor.2.5.5] an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$ such that $E = f_1$ and $3L_2 \sim f_1 + \ldots + f_{10}$.

In the case $E_1.M_2 = 3$ we get $E_1.L_2 = 4$, whence we can assume, possibly after renumbering, that $E_1.f_i = 1$ for $1 \leq i \leq 8$ and $(E_1.f_9, E_1.f_{10}) = (2, 2)$ or $(1, 3)$. In the latter case we have $(E_1 + f_{10})^2 = 6$ and $\phi(E_1 + f_{10}) = 2$, whence we can write $E_1 + f_{10} \sim A_1 + A_2 + A_3$ for some $A_i > 0$ such that $A_i^2 = 0$, $A_i.A_j = 1$ for $i \neq j$. But $f_i.(E_1 + f_{10}) = 2$ for all $1 \leq i \leq 9$, a contradiction. Hence $(E_1.f_9, E_1.f_{10}) = (2, 2)$. One easily sees that there is an isotropic divisor $f_{19} > 0$ such that $f_{19}.f_1 = f_{19}.f_9 = 2$ and $L_2 \sim f_1 + f_9 + f_{19}$. Therefore $E_1.f_{19} = 1$. Setting $E_2' = f_9$ and $E_3' = f_{19}$ we get the first case in (b-ii).

In the case $E_1.M_2 = 4$ we get $E_1.L_2 = 5$, whence we can assume, possibly after renumbering, that $E_1.f_i = 1$ for $1 \leq i \leq 5$. As above there is an isotropic divisor $f_{12} > 0$ such that $f_{12}.f_1 = f_{12}.f_2 = 2$ and $L_2 \sim f_1 + f_2 + f_{12}$. Therefore $E_1.f_{12} = 3$. Setting $E_2' = f_2$ and $E_3' = f_{12}$ we obtain the second case in (b-ii).

Finally, we have left the case with $L_2 \sim M_2$, where $\beta = \alpha$.

We have $L_1 \sim 2E_1 + M_2$, whence $(E_1.M_2)^2 = \phi(L_1)^2 \leq L_1^2 = 4E_1.M_2 + M_2^2$, so that (23) and [KL2, Prop.1] give $E_1.M_2 \leq 4$. In particular $M_2^2 \leq 6$ by (23).

If $\beta = \alpha = 3$, by definition of α, we must have $1 + E_1.M_2 = E_1.(L_1 + E) > \phi(H) = 2 + E.M_2$, whence $E_1.M_2 = 4$, $E.M_2 = 2$ and $M_2^2 = 2$ by (22). Then $E.E_2 = E.E_3 = 1$ and for $i = 2, 3$ we have $E_i.L_1 = 2E_i.E_1 + 1 \geq \phi(L_1) = E_1.M_2 = 4$, whence $E_1.E_2 = E_1.E_3 = 2$ and we get the third case in (a-iii).

In the remaining cases we have $\beta = \alpha = 2$.

If $M_2^2 = 2$ using again $\phi(L_1)^2 \leq L_1^2$, $E_i.L_1 \geq \phi(L_1)$, (22) and (23) together with $H^2 \geq 32$ or $H^2 = 28$, we deduce the possibilities $(E.M_2, E_1.M_2) = (3,3)$, $(2,4)$, $(3,4)$ or $(4,4)$. By symmetry one easily sees that one gets the cases in (a-i).

If $M_2^2 = 4$ we similarly get $(E.M_2, E_1.M_2) = (3,3)$, $(3,4)$ or $(4,4)$. From the first two cases, using Lemma 4.3 for the second, we obtain the cases in (b-i). If $(E.M_2, E_1.M_2) = (4,4)$, we now show that H also has a ladder decomposition of type (III). It will follow that S is nonextendable by Section 13.

We have $E.H = 6$, whence $(H-3E)^2 = 8$ and $H - 3E > 0$ by Lemma 4.2. If $\phi(H-3E) = 1$ we can write $H - 3E \sim 4A_1 + A_2$ with $A_i > 0$, $A_i^2 = 0$ and $A_1.A_2 = 1$. Now $6 \leq H.A_1 = 3E.A_1 + 1$ gives $E.A_1 \geq 2$, whence the contradiction $6 = H.E = 4E.A_1 + E.A_2 \geq 8$. Therefore there is an $E_1' > 0$ such that $(E_1')^2 = 0$ and $E_1'.(H-3E) = 2$. Since $(H - 3E - 2E_1')^2 = 0$, by Lemma 4.2 we can write $H \sim 3E + 2E_1' + E_2'$, with $E_2' > 0$, $(E_2')^2 = 0$ and $E_1'.E_2' = 2$. From $6 \leq H.E_1' = 3E.E_1' + 2$ we get $E.E_1' \geq 2$. Now from $6 = H.E = 2E.E_1' + E.E_2'$ we see that we cannot have $E.E_1' \geq 3$, for then $E.E_1' = 3$, $E.E_2' = 0$, but this gives $E_2' \equiv qE$ for some $q \geq 1$ by Lemma 4.2, whence the contradiction $2 = E_1'.E_2' = 3q$. Therefore $E.E_1' = 2$ and since $E_1'.L_1 = E_1'.(H-3E) + E_1'.E = 4 = \phi(L_1)$ we obtain a ladder decomposition of H of type (III), as claimed.

If $M_2^2 = 6$, by (22) and (23) we get, as above, $E_1.M_2 = E.M_2 = 4$, yielding by symmetry the case in (c) in addition to the case $(E.E_2, E.E_3, E.E_4, E_1.E_2, E_1.E_3, E_1.E_4) = (1,1,2,1,2,1)$. In the latter case we note that $\phi(H) = E.H = E_1.H = 6$ and $\phi(H - 2E_1) = \phi(2E + E_2 + E_3 + E_4) = E_3.(H - 2E_1) = 4$. Hence we can decompose H with respect to E_1 and E_3, which means that H is also of type (III) and S is nonextendable by Section 13. □

For the proof of Proposition 10.3 we will need the following:

Claim 10.2. *If $\beta \geq 5$ and either M_2 or $E_1 + M_2$ is not quasi-nef, then S is nonextendable.*

Proof. We first claim that $E_1.M_2 \geq E.M_2$.

Indeed, if $L_3 \sim E + M_2$, then by the removing conventions in Section 6, page 19, we must have $E_1.L_3 > E.L_3$, whence $E_1.M_2 \geq E.M_2$. In all other cases we have $L_1 \sim 2E_1 + \varepsilon E + M_2$, with $0 \leq \varepsilon \leq 2$, so that $\beta \geq 5$ implies that $\alpha \geq 3$. Therefore $\varepsilon + 1 + E_1.M_2 = E_1.(L_1 + E) > \phi(H) = 2 + E.M_2$, whence $E_1.M_2 > 1 - \varepsilon + E.M_2 \geq E.M_2 - 1$. This proves our assertion.

Assume now there is a $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.M_2 \leq -2$. By Lemma 4.11 we can write $M_2 \sim A + k\Delta$, where $A > 0$, $A^2 = M_2^2$, $k = -\Delta.M_2 = \Delta.A \geq 2$ and A is primitive.

If $\Delta.E_1 < 0$, then $\Delta.E > 0$ since H is ample and by Lemma 6.3 we have $E_1 \equiv E + \Delta$. But this yields $E_1.M_2 = (E + \Delta).M_2 < E.M_2$, a contradiction. Hence $\Delta.E_1 \geq 0$.

If $\Delta.E = 0$, then since H is ample, we get $\Delta.E_1 \geq 2$, whence $(E_1 + \Delta)^2 \geq 2$. Since $E.(E_1 + \Delta) = 1$, we can write $E_1 + \Delta \sim (E_1.\Delta - 1)E + E'$ for $E' > 0$ satisfying $(E')^2 = 0$ and $E.E' = 1$. From $E.M_2 = \phi(H) - 2 \geq \phi(L_1) - 2 = E_1.M_2 + \beta - \alpha - 2 \geq E_1.M_2 + \Delta.M_2 = (E_1.\Delta - 1)E.M_2 + E'.M_2$, we get $E_1.\Delta = 2$, $\Delta.M_2 = -2$, $\beta = \alpha$, $E_1.M_2 = E.M_2 + 2$ and $E'.M_2 = 0$, whence $M_2 \equiv E'$ by Lemma 4.2. It follows that $E_1 + \Delta \equiv E + M_2$, whence the absurdity $0 = (E_1 + \Delta).\Delta = (E + M_2).\Delta < 0$. Hence $\Delta.E > 0$.

Now define $B = \lfloor \frac{\beta}{2} \rfloor E + E_1 + \Delta = \lfloor \frac{\beta - 2}{2} \rfloor E + E_1 + (E + \Delta)$. Then $B^2 \geq 6$, $\phi(B) \geq 2$ and $H - 2B \sim \epsilon E + (M_2 - 2\Delta)$, with $M_2 - 2\Delta > 0$, $(M_2 - 2\Delta)^2 \geq 0$ and $\epsilon = 0$ or 1. By Proposition 5.2 we are immediately done if $B^2 \geq 8$ and if $B^2 = 6$, we only need to prove that $H^2 \geq 54$.

This is satisfied. Indeed, if $B^2 = 6$, we must have $\beta = 5$, $E.\Delta = 1$ and $E_1.\Delta = 0$. Note that $E.M_2 = E.A + k\Delta.E = E.A + k \geq 2$, and if $E.M_2 = 2$, then $k = 2, E.A = 0$ and $A \equiv E$, whence the contradiction $2 = A.\Delta = E.\Delta = 1$. Then $E.M_2 \geq 3$, whence also $E_1.M_2 \geq 3$, so that $H^2 = (5E + 2E_1 + M_2)^2 \geq 62$.

Assume similarly that there is a $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.(E_1 + M_2) \leq -2$. By what we have just proved and Lemma 6.3, we can assume that $\Delta.E_1 = \Delta.M_2 = -1$, but then we get $E_1 \equiv E + \Delta$, whence $E_1.M_2 = (E + \Delta).M_2 < E.M_2$, a contradiction. $\square$

Proposition 10.3. *Let H be of type* (I) *with $\gamma = 2$ and $M_2 > 0$. Then S is nonextendable if $\beta \geq 5$.*

Proof. By Claim 10.2 we can assume that $E_1 + M_2$ and M_2 are quasi-nef. If there exists a nodal curve Γ such that $\Gamma.(E + E_1 + M_2) < 0$, then $\Gamma.E_1 = 0$ and $\Gamma.M_2 = -1$, so that $\Gamma.E = 0$. But this yields $\Gamma.H = -1$, a contradiction. Therefore $E + E_1 + M_2$ is nef.

Set $D_0 = kE + E_1 + M_2$ with $k = \lfloor \frac{\beta-1}{2} \rfloor \geq 2$. Then $H - D_0 \sim (\beta - k)E + E_1$ and $H - D_0 - 2E = (\beta - k - 2)E + E_1$ are base-component free by Lemma 6.2.

To prove the surjectivity of μ_{V_D,ω_D} we apply Lemma 5.8. We have $h^1(D_0 + K_S - 2E) = h^1((k-2)E + E_1 + M_2 + K_S) = 0$ by Theorem 4.13 and $h^2(D_0 + K_S - 4E) = h^0((4-k)E - E_1 - M_2) = 0$ by the nefness of E.

Now $h^0(H - 2D_0 - 2E) = 0$ by the nefness of E and the exact sequence

$$0 \longrightarrow \mathcal{O}_S(H - 2D_0 - 4E) \longrightarrow \mathcal{O}_S(H - D_0 - 4E) \longrightarrow \mathcal{O}_D(H - D_0 - 4E) \longrightarrow 0,$$

shows that $h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq h^0(H - D_0 - 4E) + h^1(H - 2D_0 - 4E)$.

As $\beta - k - 4 \geq -1$ we have $h^0(H - D_0 - 4E) = h^0((\beta - k - 4)E + E_1) = \beta - k - 3$ by Lemma 6.2 and as $\beta - 2k - 4 < 0$ we have $h^1(H - 2D_0 - 4E) = h^1((\beta - 2k - 4)E - M_2) = 0$ by Theorem 4.13. Therefore (11) holds and μ_{V_D,ω_D} is surjective by Lemma 5.8.

To end the proof we deal with the Gaussian map Φ_{H_D,ω_D}. By Lemma 10.1 we can assume that $M_2^2 \leq 2$ and using Theorem 4.13 it can be easily seen that $h^0(M_2 - E) \leq 1$.

We have $D_0^2 = 2k(1+E.M_2)+2E_1.M_2+M_2^2 \geq 12$, unless $\beta = 5, 6$, $E.M_2 = E_1.M_2 = 1$ and $M_2^2 = 0$, whence $D_0^2 = 10$. Since $2D_0 - H \sim (2k-\beta)E + M_2$, the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d) unless possibly if $(E.M_2, E_1.M_2, M_2^2) = (1, 1, 0)$ and $h^0(M_2 - E) > 0$ if $\beta = 5$ or $h^0(M_2 - 2E) > 0$ if $\beta = 6$. But in the latter case we have the contradiction $2 = h^0(2E) \leq h^0(M_2) = 1$. Therefore S is nonextendable by Proposition 5.1 except possibly for the case $(E.M_2, E_1.M_2, M_2^2, \beta) = (1, 1, 0, 5)$ and $h^0(M_2 - E) > 0$.

This case will be treated in Lemma 10.4 below (where we will set $E_2 = M_2$). $\square$

Lemma 10.4. *If $H \sim 5E + 2E_1 + E_2$ with $E.E_1 = E.E_2 = E_1.E_2 = 1$ and $E_2 > E$, then S is nonextendable.*

To prove the lemma we will use the following two results:

Claim 10.5. *Set $E_0 = E$. Let $F > 0$ be a divisor such that $F^2 = 0$ and $F.E = F.E_1 = F.E_2 = 1$. If F is not nef there exists a nodal curve R such that $F \equiv E_i + R$ and $E_i.R = 1$ for some $i \in \{0, 1, 2\}$.*

Proof. Let R be a nodal curve such that $k := -F.R \geq 1$. By Lemma 4.11 we can write $F \sim A + kR$ with $A > 0$ primitive and $A^2 = 0$. Since H is ample there is an $i \in \{0, 1, 2\}$ such that $E_i.R \geq 1$. Hence $1 = E_i.F = E_i.A + kE_i.R \geq k$, so that $k = 1$, $E_i.A = 0$, $E_i.R = 1$ and $A \equiv E_i$ by Lemma 4.2. $\square$

Claim 10.6. *There is an isotropic effective* 10*-sequence $\{F_1, \ldots, F_{10}\}$ such that $F_1 = E$, $F_2 = E_1$, $F_3 = E_2$. For $4 \leq i \leq 10$ set $F'_i = E + E_1 + E_2 - F_i$. Then $F'_i > 0$, $(F'_i)^2 = 0$ and $F'_i.E = F'_i.E_1 = F'_i.E_2 = 1$. Moreover, up to renumbering $F_4, \ldots, F_{10}$, we can assume that:*

(i) *F_i is nef for $7 \leq i \leq 10$.*
(ii) *$E + F'_i$ is nef for $9 \leq i \leq 10$.*
(iii) *If $E_2 > E$ then $h^0(2F_{10} + E - E_2 + K_S) = 0$.*

Proof. First of all the 10-sequence exists since by [CD, Cor.2.5.6] we can complete the isotropic 3-sequence $\{E, E_1, E_2\}$ to an isotropic effective 10-sequence.

To see (i) suppose that $F_4, \dots, F_7$ are not nef. By Claim 10.5 there is an $i \in \{0,1,2\}$ and two indices $j,k \in \{4,\dots,7\}$, $j \neq k$ such that $F_j \equiv E_i + R_j$ and $F_k \equiv E_i + R_k$. Therefore $R_j.R_k = (F_j - E_i).(F_k - E_i) = -1$, a contradiction. Upon renumbering we can assume that F_i is nef for $7 \le i \le 10$.

Now the definition of F'_i easily gives that $(F'_i)^2 = 0$ and $F'_i.E = F'_i.E_1 = F'_i.E_2 = 1$. Also $F'_i > 0$ by Riemann-Roch since $F'_i.E = 1$ implies that $H^2(F'_i) = 0$. To see (ii) suppose that $E + F'_7$, $E + F'_8$ and $E + F'_9$ are not nef. By Claim 10.5 there is an $i \in \{1,2\}$ and two indices $j,k \in \{7,8,9\}$, $j \neq k$ such that $F'_j \equiv E_i + R_j$ and $F'_k \equiv E_i + R_k$, giving a contradiction as above. Upon renumbering we can assume that $E + F'_i$ is nef for $9 \le i \le 10$.

To see (iii) let F be either F_9 or F_{10} and suppose that $2F + E - E_2 + K_S \ge 0$. Let Γ be a nodal component of $E_2 - E$. Since $2F + K_S \ge E_2 - E \ge \Gamma$ and $h^0(2F + K_S) = 1$, we get that Γ must be either a component of F or of $F + K_S$. Therefore Γ is, for example, a component of both F_9 and F_{10} and this is not possible since $F_9.F_{10} = 1$ and they are both nef and primitive. □

Proof of Lemma 10.4. By Claim 10.6(ii) we know that $E + F'_{10} = 2E + E_1 + E_2 - F_{10}$ is nef, whence, using [CD, Prop.3.1.6 and Cor.3.1.4], we can choose $F \equiv F_{10}$ so that, setting $F' = E + E_1 + E_2 - F$, we have that $E + F' = 2E + E_1 + E_2 - F$ is a base-component free pencil. Let $D_0 = 3E + E_1 + F$. Then $D_0^2 = 14$, $\phi(D_0) = 2$ and D_0 is nef by Lemma 6.2 and Claim 10.6(i). Now $H - D_0 = 2E + E_1 + E_2 - F = E + F'$ is a base-component free pencil. Also $2D_0 - H = 2F + E - E_2$ so that $E.(2D_0 - H) = 1$ and $F.(2D_0 - H) = 0$. If $h^0(2D_0 - H) \ge 2$ we can write $2D_0 - H \sim G + M$ where G is the base component and M is base-component free. Since both E and F are nef we get that $E.M \le 1$ and $F.M = 0$. The latter implies that $M^2 = 0$ whence $M \sim 2hF$ for some $h \ge 1$ by [CD, Prop.3.1.4], but then we get the contradiction $E.M \ge 2$. Therefore $h^0(2D_0 - H) \le 1$ and Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d).

Now $E.(H - 2D_0) = -1$ whence $h^0(H - 2D_0) = 0$. Also $(H - 2D_0)^2 = -2$ whence, by Riemann-Roch, $h^1(H - 2D_0) = h^2(H - 2D_0) = h^0(2D_0 - H + K_S) = h^0(2F + E - E_2 + K_S) = h^0(2F_{10} + E - E_2 + K_S) = 0$ by Claim 10.6(iii).

Therefore μ_{V_D,ω_D} is surjective by (15) and S is nonextendable by Proposition 5.1. □

11. Remaining cases in (I) with $\gamma = 2$ and $M_2 > 0$

As the cases left have $\beta \le 4$ by Proposition 10.3, we have, for the whole section,

$$H \sim \beta E + 2E_1 + M_2, \text{ with } \beta = 2,\ 3 \text{ or } 4,$$

and either $M_2^2 = 0$ or we are in one of the cases of Lemma 10.1.

11.1. **The case $M_2^2 = 0$.** We write $M_2 = E_2$ for a primitive $E_2 > 0$ with $E_2^2 = 0$.

11.1.1. $\beta = 2$. From (22) and (23) we get $1 \le E.E_2 \le E_1.E_2 \le E.E_2 + 2$. Moreover, since $L_1 \sim 2E_1 + E_2$, we get $(\phi(L_1))^2 = (E_1.E_2)^2 \le L_1^2 = 4E_1.E_2$, whence $E_1.E_2 \le 3$ by [KL2, Prop.1], as E_2 is primitive. Since $H^2 \ge 28$, we are left with the cases $(E.E_2, E_1.E_2) = (2,3)$ or $(3,3)$, so that S is nonextendable by Lemma 5.6(iii-b).

11.1.2. $\beta = 3$. From (22) and (23) we get $1 \le E.E_2 \le E_1.E_2 + 1 \le E.E_2 + \alpha$.

If $\alpha = 2$ we get $E.E_2 - 1 \le E_1.E_2 \le E.E_2 + 1$. Moreover, since $L_2 \sim E + E_2$ is of small type, we must have $E.E_2 \le 3$ or $E.E_2 = 5$. Furthermore, since $L_1 \sim E + 2E_1 + E_2$, we get $(\phi(L_1))^2 = (1 + E_1.E_2)^2 \le L_1^2 = 4 + 4E_1.E_2 + 2E.E_2$. However, in the case $(E.E_2, E_1.E_2) = (3,4)$, we find $(L_1^2, \phi(L_1)) = (26,5)$, which is impossible by [KL2, Prop.1]. This yields that $E.E_2 = 2,3,5$ if $E_1.E_2 = E.E_2 - 1$; $E.E_2 = 1,2,3$ if $E_1.E_2 = E.E_2$; and $E.E_2 = 1,2$ if $E_1.E_2 = E.E_2 + 1$.

If $\alpha = 3$ we must have, by (13), that $E_1.(H-3E) = \phi(H)$, whence $E_1.E_2 = 2 + E.E_2$. Moreover, since $L_1 \sim 2E_1 + E_2$, we get $(\phi(L_1))^2 = (E_1.E_2)^2 \leq L_1^2 = 4E_1.E_2$, whence $E_1.E_2 \leq 3$ by [KL2, Prop.1] since E_2 is primitive. Hence $E_1.E_2 = 3$ and $E.E_2 = 1$.

To summarize, using $H^2 \geq 32$ or $H^2 = 28$, we have the following cases:

$$\begin{array}{lll} E_1.E_2 = E.E_2 - 1, & E.E_2 = 2,\ 3 \text{ or } 5, & g = 15,\ 20 \text{ or } 30. \\ E_1.E_2 = E.E_2, & E.E_2 = 2 \text{ or } 3, & g = 17 \text{ or } 22. \\ E_1.E_2 = 3, & E.E_2 = 2, & g = 19. \end{array} \tag{25}$$

We will now show, in Lemmas 11.1-11.4, that S is nonextendable in the five cases of genus $g \geq 17$. The case with $g = 15$ is case (b1) in the proof of Proposition 15.1.

Lemma 11.1. *Let $H \sim 3E + 2E_1 + E_2$ be as in (25) with $(E.E_2, E_1.E_2, g) = (5, 4, 30)$. Then S is nonextendable.*

Proof. We have $H^2 = 58$ and $\phi(H) = E.H = E_1.H = 7$. Hence both E and E_1 are nef by Lemma 4.14. Let now $H' = H - 4E$. Then $(H')^2 = 2$ and consequently we can write $H \sim 4E + A_1 + A_2$ for $A_i > 0$ primitive with $A_i^2 = 0$ and $A_1.A_2 = 1$. Since $E.H = E.A_1 + E.A_2 = 7$ we can assume by symmetry that either (a) $(E.A_1, E.A_2) = (2, 5)$ or (b) $(E.A_1, E.A_2) = (3, 4)$. Also since $E_1.H = 7$ we have $E_1.(A_1 + A_2) = 3$, whence we have the two possibilities $(E_1.A_1, E_1.A_2) = (2, 1)$ or $(1, 2)$.

In case (b) we get $A_1.H = 13$, whence $(H - 2(E + A_1))^2 = 2$. Since $(H - 2(E + A_1)).E = 1$, we have $H - 2(E + A_1) > 0$ by Riemann-Roch, whence S is nonextendable by Proposition 5.2.

In case (a) we get $A_1.H = 9$. Now if $E_1.A_1 = 2$, we get $(H - 2(E + A_1 + E_1))^2 = 6$, and as above S is nonextendable by Proposition 5.2.

Hence the only case left is (a) with $(E_1.A_1, E_1.A_2) = (1, 2)$. Note that $E_1.(H - 2E) = A_1.(H - 2E) = 5$, whence $L_1 \sim H - 2E$ and $\phi(L_1) = A_1.L_1 = 5$. Therefore we can continue the decomposition with respect to A_1 instead of E_1. Since H now is also of type (III), S is nonextendable by Sections 6 and 13. □

Claim 11.2. *Let $H \sim 3E + 2E_1 + E_2$ be as in (25) with $(E.E_2, E_1.E_2, g) = (3, 2, 20)$ (respectively $(E.E_2, E_1.E_2, g) = (3, 3, 22)$). Then there exists an isotropic effective 5-sequence $\{E, F_1, F_2, F_3, F_4\}$ (respectively an isotropic effective 4-sequence $\{E, F_1, F_2, F_3\}$ together with an isotropic divisor $F_4 > 0$ such that $E.F_4 = F_2.F_4 = F_3.F_4 = 1$ and $F_1.F_4 = 2$) such that $H \sim 2E + 2F_1 + F_2 + F_3 + F_4$ and:*

- (a) *F_1 is nef and F_i is quasi-nef for $i = 2, 3, 4$;*
- (b) *$|E + F_2|$ and $|F_1 + F_3|$ are without base components;*
- (c) *$|E + F_1 + F_2 + F_3|$ and $|E + F_1 + F_4|$ are base-point free;*
- (d) *$h^1(F_1 + F_4 - F_2) = h^2(F_1 + F_4 - F_2) = 0$.*

Proof. Since $(E + E_2)^2 = 6$ and both E and E_2 are primitive, we can write $E + E_2 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i.A_j = 1$ for $i \neq j$. We easily find (possibly after renumbering) that $A_i.E = A_i.E_2 = A_1.E_1 = A_2.E_1 = 1$ for $i = 1, 2, 3$ and $A_3.E_1 = 1$ if $g = 20$ and 2 if $g = 22$. Moreover $A_i.H \leq 8 < 2\phi(H) = 10$, whence all the A_i's are quasi-nef by Lemma 4.14.

Assume now there is a nodal curve R_i with $R_i.A_i = -1$ for $(i, g) \neq (3, 22)$. Then we can as usual write $A_i \sim B_i + R_i$, with $B_i > 0$ primitive and isotropic. Since $A_i.H = 6$ we deduce that $B_i \equiv E$ or $B_i \equiv E_1$, where the latter case only occurs if $g = 20$.

If $g = 20$, then, since for $i \neq j$, we have $(E + R_i).(E + R_j) = 2 + R_i.R_j = (E_1 + R_i).(E_1 + R_j)$, we see that at most two of the A_i's can be not nef, otherwise we would get $R_i.R_j = -1$, a contradiction. Possibly after reordering the A_i's and adding K_S to two of them, we can therefore assume that A_1 is nef, and that either A_2 is nef or $A_2 \sim E + R + K_S$ for R a nodal curve with $E.R = 1$. Now E_1 is nef, by Lemma 4.14, as $E_1.H = \phi(H) = 5$, so that both $|E_1 + A_1|$ and $|E + A_2|$ are without fixed components. Setting $F_1 = E_1$, $F_2 = A_2$, $F_3 = A_1$ and $F_4 = A_3$ we therefore have the desired

decomposition satisfying (a) and (b). It also follows by construction that $E+F_1+F_2+F_3$ and $E+F_1+F_4$ are nef, the latter because E and F_1 are, and F_4 is either nef or $F_4 \equiv A+R'$ with $A=E$ or $A=E_1$, for R' a nodal curve with $A.R'=1$. Therefore (c) also follows.

If $g=22$, we similarly find that we can assume that A_1 and A_2 are nef. Moreover $A_1.L_1 = A_1.(H-2E) = E_1.(H-2E) = 4$, so if E_1 is not nef, we can substitute E_1 with A_1 and repeat the process. Therefore we can assume that E_1 is nef as well. Again both $|E_1+A_1|$ and $|E+A_2|$ are without fixed components, and setting $F_1=E_1$, $F_2=A_2$, $F_3=A_1$ and $F_4=A_3$ we therefore have the desired decomposition satisfying (a) and (b). Now $E+F_1+F_2+F_3$ is again nef by construction. To see that $E+F_1+F_4$ is nef, assume, to get a contradiction, that there is a nodal curve Γ with $\Gamma.(E+F_1+F_4)<0$. Then $\Gamma.F_4=-1$ and $\Gamma.(E+F_1)=0$ by (a). The ampleness of H implies $\Gamma.(F_2+F_3)\geq 2$, whence the contradiction $(F_4-\Gamma)^2=0$ and $(F_4-\Gamma).(F_2+F_3)\leq 0$, recalling that $F_4-\Gamma>0$ by Lemma 4.11. Therefore (c) is proved.

We now prove (d).

If $g=20$ then $(F_1+F_4-F_2)^2=-2$ and $(F_1+F_4-F_2).H=5=\phi(H)$, whence $h^2(F_1+F_4-F_2)=0$ and if $F_1+F_4-F_2>0$ it is a nodal cycle, so that either $h^0(F_1+F_4-F_2)=0$ or $h^0(F_1+F_4-F_2+K_S)=0$. Replacing F_1 with F_1+K_S if necessary, we can arrange that $h^0(F_1+F_4-F_2)=0$, whence also $h^1(F_1+F_4-F_2)=0$ by Riemann-Roch.

If $g=22$, then $(F_1+F_4-F_2)^2=0$ and $(F_1+F_4-F_2).H=8<2\phi(H)$, whence (d) follows by Lemma 4.14 and Theorem 4.13. □

Lemma 11.3. *Let $H\sim 3E+2E_1+E_2$ be as in* (25) *with* $(E.E_2, E_1.E_2, g) = (3,2,20)$ *or* $(3,3,22)$. *Then S is nonextendable.*

Proof. By Claim 11.2 we can choose $D_0 = E+F_1+F_2+F_3$ with $D_0^2=12$ and both D_0 and $H-D_0 \sim E+F_1+F_4$ base-point free.

We have $h^0(2D_0-H)=h^0(F_2+F_3-F_4)\leq 1$ by Lemma 4.14, as $(F_2+F_3-F_4).H\leq 6<2\phi(H)$. Hence the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d).

To show the surjectivity of μ_{V_D,ω_D} we use Claim 11.2(b) and let $D_1\in|E+F_2|$ and $D_2\in|F_1+F_3|$ be general smooth curves and apply Lemma 5.7.

Now $H-D_0-D_1\sim F_1+F_4-F_2$ whence $h^1(H-D_0-D_1)=0$ by Claim 11.2(d), so that $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (16) since $(H-D_0).D_1=(E+F_1+F_4).(E+F_2)=5$.

Since $(H-D_0-D_2).H=(E+F_4-F_3).H\leq 7<2\phi(H)$ we have that $h^0(H-D_0-D_2)\leq 1$ by Lemma 4.14 and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18).

Therefore μ_{V_D,ω_D} is surjective whence S is nonextendable by Proposition 5.1. □

Lemma 11.4. *Let $H\sim 3E+2E_1+E_2$ be as in* (25) *with $E.E_2=2$ and* $(E_1.E_2,g)=(2,17)$ *or* $(3,19)$. *Then S is nonextendable.*

Proof. We first observe that it is enough to find an isotropic divisor $F>0$ such that $E.F=1$, $F.H=6$ if $g=17$ and $F.H=7$ if $g=19$ and $B:=E+F$ is nef.

In fact the latter implies that $H\sim 2B+A$, with $A>0$ isotropic with $E.A=2$ and $F.A=4$ if $g=17$ and $F.A=5$ if $g=19$. As we assume that H is not 2-divisible in $\operatorname{Num} S$, A is automatically primitive and it follows that S is nonextendable by Lemma 5.6(iii-b).

To find the desired F we first consider the case $g=17$.

Set $Q=E+E_1+E_2$. Then $Q^2=10$ and $\phi(Q)=3$. By [CD, Cor.2.5.5] there is an isotropic effective 10-sequence $\{f_1,\ldots,f_{10}\}$ with

$$3Q\sim f_1+\ldots+f_{10}.$$

Since $E.Q=E_1.Q=3$, we can assume that $f_1=E$ and $f_2=E_1$ and then $E_2.f_i=1$ for $i\geq 3$.

We now claim that $E+f_i$ is not nef for at most one $i\in\{3,\ldots,10\}$.

Indeed, note that, for $i \geq 3$, we have $f_i.H = 6 < 2\phi(H) = 8$, whence each f_i is quasi-nef by Lemma 4.14. Now assume that $R_i.(E + f_i) < 0$ for some nodal curve R_i. Then $R_i.E = 0$ and $R_i.f_i = -1$, so that $f_i \sim \overline{f_i} + R_i$, by Lemma 4.11, with $\overline{f_i} > 0$ primitive and $\overline{f_i}^2 = 0$. Since H is ample we must have $R_i.E_j > 0$ for $j = 1$ or 2.

If $R_i.E_2 > 0$ then $E_2.f_i = 1$ implies $\overline{f_i} \equiv E_2$ and $R_i.E_2 = 1$. But then we get the contradiction $E.f_i = E.(E_2 + R_i) = 2$.

Therefore $R_i.E_1 > 0$, so that $\overline{f_i} \equiv E_1$ and $R_i.E_1 = 1$.

Now suppose that also $E+f_j$ is not nef for $j \in \{3, \ldots, 10\} - \{i\}$. Then $R_i.R_j = (f_i - E_1).(f_j - E_1) = -1$, a contradiction. Therefore $E + f_i$ is not nef for at most one $i \in \{3, \ldots, 10\}$.

Now one easily verifies that any $F \in \{f_3, \ldots, f_{10}\}$ such that $E + F$ is nef satisfies the desired numerical conditions.

We next consider the case $g = 19$.

Since $(E_1 + E_2)^2 = 6$ and $\phi(E_1 + E_2) = 2$ we can find an isotropic effective 3-sequence $\{f_3, f_4, f_5\}$ such that $E_1 + E_2 \sim f_3 + f_4 + f_5$. Since $E.(E_1 + E_2) = E_1.(E_1 + E_2) = 3$ we have $f_i.E = f_i.E_1 = 1$ for $i = 3, 4, 5$, so that we have an isotropic effective 5-sequence $\{f_1, \ldots, f_5\}$ with $f_1 = E$ and $f_2 = E_1$ such that $H \sim 3f_1 + f_2 + f_3 + f_4 + f_5$. By [CD, Cor.2.5.6] we can complete the sequence to an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$. Note that for $i \geq 6$ we have $f_i.H = 7 < 2\phi(H) = 8$, whence each f_i is quasi-nef by Lemma 4.14.

Now the same arguments as above can be used to prove that $E + f_i$ is nef for at least one $i \in \{6, \ldots, 10\}$, whence any $F \in \{f_6, \ldots, f_{10}\}$ such that $E + F$ is nef satisfies the desired numerical conditions. $\square$

11.1.3. $\beta = 4$. From (22) and (23) we get $1 \leq E.E_2 \leq E_1.E_2 + 2 \leq E.E_2 + \alpha$.

If $\alpha = 2$ we get $E.E_2 - 2 \leq E_1.E_2 \leq E.E_2$. Moreover, since $L_2 \sim 2E + E_2$ is not of small type, we get $(\phi(L_2))^2 = (E.E_2)^2 \leq L_2^2 = 4E.E_2$, whence $E.E_2 \leq 3$ by [KL2, Prop.1]. Therefore $(E.E_2, E_1.E_2) \in \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$. The first case is case (b2) in the proof of Proposition 15.1 and in the other cases S is nonextendable by Lemmas 6.2 and 5.6(iii-a).

If $\alpha = 3$ or 4 we must have $E_1.(H - \alpha E) = \phi(H)$ by (13), whence $E_1.E_2 = E.E_2 + \alpha - 2$. Moreover $L_1 \sim (4-\alpha)E + 2E_1 + E_2$ and using $(\phi(L_1))^2 \leq L_1^2$, we get $E_1.E_2 \leq 4$. If equality holds then $(L_1^2, \phi(L_1)) = (26, 5)$ or $(16, 4)$, both excluded by [KL2, Prop.1], as E_2 is primitive. Therefore $(E.E_2, E_1.E_2) = (1,2), (1,3)$ or $(2,3)$ and S is nonextendable by Lemmas 6.2 and 5.6(iii-a).

11.2. **The case $M_2^2 = 2$.** We write $M_2 = E_2 + E_3$ for primitive $E_2 > 0$ and $E_3 > 0$ with $E_2^2 = E_3^2 = 0$ and $E_2.E_3 = 1$, as in Lemma 10.1(a).

11.2.1. $\beta = 2$. By Lemma 10.1 we have left to treat the cases (a-i), that is

$$(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1,2,1,2),\ (1,2,2,1),\ (1,1,2,2),\ (2,2,2,2),\ (1,2,2,2). \tag{26}$$

We first show that S is nonextendable in the first case of (26).

Since $E_2.H = \phi(H) = 5$ and $E_3.H = 9 < 2\phi(H)$ we have that E_2 is nef and E_3 is quasi-nef by Lemma 4.14. In particular we get that $h^1(E_2 + E_3) = h^1(E_2 + E_3 + K_S) = 0$ by Theorem 4.13 and $h^0(E_2 + E_3) = 2$ by Riemann-Roch. Now $D_0 := E + E_1 + E_2 + E_3$ is nef by Lemma 6.4(b) with $\phi(D_0) = 3$ and $D_0^2 = 16$. Also $H - D_0 \sim E + E_1$ is base-component free and $2D_0 - H \sim E_2 + E_3$. Then $h^0(2D_0 - H) = 2$ and $h^1(H - 2D_0) = 0$, so that μ_{V_D, ω_D} is surjective by (15) and Φ_{H_D, ω_D} is surjective by Theorem 5.3(e), as $\operatorname{gon}(D) = 6$ by [KL2, Cor.1], whence $\operatorname{Cliff}(D) = 4$, as D has genus 9 [ELMS, §5]. By Proposition 5.1, S is nonextendable.

We next show that S is nonextendable in the last four cases in (26).

By Lemmas 4.14 and 6.4(b) we see that E_2 and E_3 are quasi-nef and $E + E_1 + E_2$ and $E + E_1 + E_3$ are base-point free.

Now set $D_0 = E + E_1 + E_2$. Then $D_0^2 \geq 8$, D_0 is nef, $\phi(D_0) \geq 2$ and $H - D_0 \sim E + E_1 + E_3$ is base-point free. Moreover $h^0(2D_0 - H) = 0$ as $(2D_0 - H).H = (E_2 - E_3).H \leq 0$, so that Φ_{H_D,ω_D} is surjective by Theorem 5.3(c). Now, in all cases except for $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 2, 2)$, we have $(H - 2D_0)^2 = -2$ and $(H - 2D_0).H = 0$, so that $h^0(H - 2D_0) = h^2(H - 2D_0) = 0$, whence $h^1(H - 2D_0) = 0$ by Riemann-Roch and μ_{V_D,ω_D} is surjective by (14) (noting that $(H - D_0)^2 = 10$ in the case $(2, 2, 2, 2)$, while $H - D_0$ is not 2-divisible in $\operatorname{Pic} S$ as either $E.(H - D_0) = 3$ or $E_1.(H - D_0) = 3$ in the other two cases). By Proposition 5.1, S is nonextendable in those cases.

We now prove the surjectivity of μ_{V_D,ω_D} in the case $(E.E_2, E.E_3, E_1.E_2, E_1.E_3) = (1, 2, 2, 2)$.

Note that $E_1 + E_2$ is nef by Lemma 6.4(e), whence base-point free, and that $E_1 + E_3$ is quasi-nef. To see the latter, let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.E_1 + \Delta.E_3 \leq -2$. As E_1 is quasi-nef by Lemma 6.3 and E_3 is quasi-nef we get, again by Lemma 6.3, that $\Delta.E_1 = \Delta.E_3 = -1$ and $E_1 \equiv E + \Delta$, giving the contradiction $\Delta.E_3 = 0$. Hence $E_1 + E_3$ is quasi-nef.

To show the surjectivity of μ_{V_D,ω_D} we let $D_1 = E$ and $D_2 \in |E_1 + E_2|$ be a general smooth curve and apply Lemma 5.7. The map $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (17) since $h^1(H - D_0 - D_1) = h^1(E_1 + E_3) = 0$ by Theorem 4.13. Finally, $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18), using the fact that $h^0(H - D_0 - D_2) = h^0(E + E_3 - E_2) \leq 1$ by Lemma 4.14, as $(E + E_3 - E_2).H = 7 < 2\phi(H)$. Therefore μ_{V_D,ω_D} is surjective and S is nonextendable by Proposition 5.1.

11.2.2. $\beta = 3, 4$. By Lemma 10.1 we have left to treat the cases (a-ii) and (a-iii), that is

$$\beta = 3, \quad (E.E_2, E.E_3, E_1.E_2, E_1.E_3, E_2.E_3) = (2, 2, 2, 2, 1), \tag{27}$$

$$\beta = 3, \quad (E.E_2, E.E_3, E_1.E_2, E_1.E_3, E_2.E_3) = (2, 2, 1, 2, 1), \tag{28}$$

$$\beta = 3, 4, \quad (E.E_2, E.E_3, E_1.E_2, E_1.E_3, E_2.E_3) = (1, 1, 2, 2, 1), \tag{29}$$

$$\beta = 3, 4, \quad (E.E_2, E.E_3, E_1.E_2, E_1.E_3, E_2.E_3) = (1, 1, 1, 2, 1), \tag{30}$$

$$\beta = 3, 4, \quad (E.E_2, E.E_3, E_1.E_2, E_1.E_3, E_2.E_3) = (1, 1, 1, 1, 1). \tag{31}$$

Claim 11.5. *In the cases* (27)-(31) *both E_2 and E_3 are quasi-nef.*

Proof. We first prove that E_2 is quasi-nef. Assume, to get a contradiction, that there exists a $\Delta > 0$ satisfying $\Delta^2 = -2$ and $\Delta.E_2 \leq -2$. Write $E_2 \sim A + k\Delta$, for $A > 0$ primitive with $A^2 = 0$ and $k = -\Delta.E_2 = \Delta.A \geq 2$. From $E_2.E_3 = 1$ it follows that $\Delta.E_3 \leq 0$. If $\Delta.E > 0$, we get from $2 \geq E.E_2 = E.A + kE.\Delta$ that $E.E_2 = k = 2$, $E.\Delta = 1$ and $E.A = 0$, whence the contradiction $E \equiv A$. Hence $\Delta.E = 0$ and the ampleness of H gives $\Delta.E_1 \geq 2$, whence the contradiction $E_1.E_2 = E_1.A + kE_1.\Delta \geq 4$. Hence E_2 is quasi-nef. The same reasoning works for E_3. □

Lemma 11.6. *In the cases* (27)-(29) *and in the cases* (30)-(31) *with $\beta = 4$ we have that S is nonextendable.*

Proof. Define $D_0 = 2E + E_1 + E_2$, which is nef by Lemma 6.4(a) with $\phi(D_0) \geq 2$ and $D_0^2 \geq 12$ in cases (27)-(29) and $D_0^2 = 10$ in cases (30) and (31).

Also $H - D_0 \sim (\beta - 2)E + E_1 + E_3$, whence $\phi(H - D_0) \geq 2$ and $H - D_0$ is base-point free by Lemma 6.4(b).

We have $2D_0 - H \sim (4 - \beta)E + E_2 - E_3$, whence $h^0(2D_0 - H) \leq 1$ in the cases (27)-(29), as $(2D_0 - H).H \leq \phi(H)$, and $h^0(2D_0 - H) = 0$ in cases (30)-(31), as $(2D_0 - H).H \leq 0$. It follows from Theorem 5.3(c)-(d) that the map Φ_{H_D,ω_D} is surjective.

We next note that μ_{V_D,ω_D} is surjective by (14) if $h^1(H - 2D_0) = h^1(E_3 - (4 - \beta)E - E_2) = 0$.

Since $(E_3 - E_2).H = 0$ in cases (29) and (31) we have $h^0(E_3 - E_2) = h^2(E_3 - E_2) = 0$, whence $h^1(E_3 - E_2) = 0$ by Riemann-Roch. It follows that μ_{V_D,ω_D} is surjective, whence S is nonextendable by Proposition 5.1 in cases (29) and (31) with $\beta = 4$. In the remaining cases we can assume that

$$h^1(E_3 - (4 - \beta)E - E_2) > 0. \tag{32}$$

We next show that μ_{V_D,ω_D} is surjective in case (28). For this we use Lemmas 5.7, 6.2 and 6.4(c) and let $D_1 \in |E+E_1|$ and $D_2 \in |E+E_2|$ be general smooth members.

By Claim 11.5 and Theorem 4.13 we have that $h^1(H-D_0-D_1) = h^1(E_3) = 0$, whence $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (16). Furthermore $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18), where one uses that $h^0(H-D_0-D_2) = h^0(E_1+E_3-E_2) \leq 1$ by Lemma 4.14 since $(E_1+E_3-E_2).H < 2\phi(H)$. Hence μ_{V_D,ω_D} is surjective and S is nonextendable by Proposition 5.1.

Finally we treat the cases (27), (29) (with $\beta = 3$) and (30) (with $\beta = 4$). Since $(E_3-(4-\beta)E-E_2)^2 = -2$ and $(E_3-(4-\beta)E-E_2).H = -\phi(H)$ in (27) and (29) (respectively 2 in (30)), we see that Riemann-Roch and (32) imply that $E+E_2-E_3+K_S$ is a nodal cycle in (27) and (29) and E_3-E_2 is a nodal cycle in (30). With β as above, it follows that

$$h^i(E+E_2-E_3) = 0 \text{ in (27) and (29) and } h^i(E_3-E_2+K_S) = 0 \text{ in (30)}, \ i = 0,1,2. \tag{33}$$

We now choose a new $D_0 := (\beta-2)E+E_1+E_3$, which is nef with $\phi(D_0) \geq 2$ and with $H-D_0$ base-point free by Lemma 6.4(a) and (b). Then $D_0^2 \geq 8$ with $h^0(2D_0-H) = h^0(E_3-E-E_2) = 0$ in (27) and (29) and $D_0^2 = 12$ with $h^0(2D_0-H) = h^0(E_3-E_2) = 1$ in (30), whence Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d).

Now (33) implies $h^1(H-2D_0) = 0$, so that μ_{V_D,ω_D} is surjective by (14) and S is nonextendable by Proposition 5.1. □

We have left the cases (30) and (31) with $\beta = 3$, which we treat in Lemmas 11.7 and 11.9.

Lemma 11.7. *Suppose $H \sim 3E+2E_1+E_2+E_3$ with $E.E_2 = E.E_3 = E_1.E_2 = E_2.E_3 = 1$, $E_1.E_3 = 2$ (the case (30) with $\beta = 3$). Then S is nonextendable.*

Proof. Since $E_2.H = 6$ one easily finds another ladder decomposition

$$H \sim 3E+2E_2+E_1+E_3', \text{ with } E_2.E_3' = 2, \tag{34}$$

and all other intersections equal to one.

We first claim that either E_1 or E_2 is nef.

In fact $\phi(L_1) = E_1.L_1 = E_1.(E+2E_1+E_2+E_3) = 4 = E_2.L_1$. By Lemma 6.3, if neither E_1 nor E_2 are nef, there are two nodal curves R_1 and R_2 such that $R_i.E = 1$ and $E_i \equiv E+R_i$, for $i = 1,2$. But then we get the absurdity $R_1.R_2 = (E_1-E).(E_2-E) = -1$.

By (34) we can and will from now on assume that we have a ladder decomposition $H \sim 3E+2E_1+E_2+E_3$ with E_2 nef.

Claim 11.8. *Either $h^0(E+E_3-E_2+K_S) = 0$, or $h^0(E+E_2-E_3) = 0$, or $h^0(E_2+E_3-E+K_S) = 0$.*

Proof. Let $\Delta_1 := E+E_3-E_2+K_S$, $\Delta_2 := E+E_2-E_3$ and $\Delta_3 := E_2+E_3-E+K_S$. Assume, to get a contradiction, that $\Delta_i \geq 0$ for all $i = 1,2,3$. Since $\Delta_i^2 = -2$ we get that $\Delta_i > 0$ for all $i = 1,2,3$.

We have $\Delta_2 \sim 2E+K_S-\Delta_1$. Since $\Delta_1.H = 6$ and $E.H = 4$, we can neither have $\Delta_1 \leq E$ nor $\Delta_1 \leq E+K_S$. Therefore, as E and $E+K_S$ have no common components, we must have $\Delta_1 = \Delta_{11}+\Delta_{12}$ with $0 < \Delta_{11} \leq E$ and $0 < \Delta_{12} \leq E+K_S$ and $\Delta_{11}.\Delta_{12} = 0$. Moreover we have $E.\Delta_{11} = E.\Delta_{12} = 0$, whence $\Delta_{1i}^2 \leq 0$ for $i = 1,2$. From $-2 = \Delta_1^2 = \Delta_{11}^2+\Delta_{12}^2$ we must have $\Delta_{1i}^2 = 0$ either for $i = 1$ or for $i = 2$. By symmetry we can assume that $\Delta_{11}^2 = 0$. Therefore $\Delta_{11} \equiv qE$ for some $q \geq 1$ by Lemma 4.2, but $\Delta_{11} \leq E$, whence $\Delta_{11} = E$ and $\Delta_{12}^2 = -2$. Moreover $\Delta_{12}.H = 2$.

Now since $E+\Delta_{12} \equiv \Delta_1 \equiv E+E_3-E_2$, we get $E_3 \equiv E_2+\Delta_{12}$ and $E_2.\Delta_{12} = 1$. Hence $\Delta_3 \sim E_2+E_3-E+K_S \sim (E+E_3+K_S-\Delta_1)+E_3-E+K_S \sim 2E_3-\Delta_1 \sim 2(E_2+\Delta_{12})-\Delta_1 \sim 2E_2+\Delta_{12}-\Delta_{11}$, therefore

$$\Delta_{11}+\Delta_3 \in |2E_2+\Delta_{12}|. \tag{35}$$

We claim that $|2E_2 + \Delta_{12}| = |2E_2| + \Delta_{12}$. To see the latter observe that it certainly holds if Δ_{12} is irreducible, for then it is a nodal curve with $E_2.\Delta_{12} = 1$ (recall that $|2E_2|$ is a genus one pencil). On the other hand if Δ_{12} is reducible then, using $\Delta_{12}.H = 2$ and the ampleness of H we deduce that $\Delta_{12} = R_1 + R_2$ where R_1, R_2 are two nodal curves with $R_1.R_2 = 1$. Moreover the nefness of E_2 allows us to assume that $E_2.R_1 = 1$ and $E_2.R_2 = 0$. But then $R_2.(2E_2 + \Delta_{12}) = -1$ so that R_2 is a base-component of $|2E_2+\Delta_{12}|$ and of course R_1 is a base-component of $|2E_2+\Delta_{12}-R_2| = |2E_2+R_1|$ and the claim is proved.

Since Δ_{11} and Δ_{12} have no common components we deduce from (35) that each irreducible component of $E = \Delta_{11}$ must lie in some element of $|2E_2|$. The latter cannot hold if E is irreducible for then we would have that $2E_2 - E > 0$ and $(2E_2 - E).E_2 = -1$ would contradict the nefness of E_2. Therefore, as is well-known, we have that $E = R_1 + \ldots + R_n$ is a cycle of nodal curves and we can assume, without loss of generality, that $E_2.R_1 = 1$ and $E_2.R_i = 0$ for $2 \leq i \leq n$. As we said above, we have $2E_2 - R_1 > 0$. Now for $2 \leq i \leq n-1$ we get $R_i.(2E_2 - R_1 - \ldots - R_{i-1}) = -1$, whence $2E_2-R_1-\ldots-R_i > 0$. Therefore $2E_2-R_1-\ldots-R_{n-1} > 0$ and since $R_n.(2E_2-R_1-\ldots-R_{n-1}) = -2$ we deduce that $2E_2 - E > 0$, again a contradiction. □

Conclusion of the proof of Lemma 11.7. We divide the proof into the three cases of Claim 11.8.

Case A: $h^0(E + E_3 - E_2 + K_S) = 0$**.** Set $D_0 = 2E + E_1 + E_3$. Then $D_0^2 = 12$ and $\phi(D_0) = 2$. Moreover D_0 is nef by Claim 11.5 and Lemma 6.4(a) and $H-D_0 \sim E+E_1+E_2$ is nef since $E+E_1$ and E_2 are (the first by Lemma 6.2), so that $|H-D_0|$ is base-point free, since $\phi(H-D_0) = E.(H-D_0) = 2$.

We have $2D_0-H \sim E+E_3-E_2$ and since $(2D_0-H).H = 6 < 2\phi(H) = 8$, we have $h^0(2D_0-H) \leq 1$ by Lemma 4.14, so that Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d).

Clearly $h^0(H-2D_0) = 0$ and we also have $h^2(H-2D_0) = h^0(2D_0-H+K_S) = h^0(E+E_3-E_2+K_S) = 0$ by assumption. Therefore $h^1(H-2D_0) = 0$ by Riemann-Roch and μ_{V_D,ω_D} is surjective by (14). Hence S is nonextendable by Proposition 5.1.

Case B: $h^0(E + E_2 - E_3) = 0$**.** We set $D_0 = E + E_1 + E_3$, so that $D_0^2 = 8$, $\phi(D_0) = 2$ and both D_0 and $H-D_0 \sim 2E+E_1+E_2$ are nef by Claim 11.5 and Lemma 6.4(a) and (b), whence base-point free. Since $2D_0 - H \sim E_3 - E - E_2$ and $(E_3 - E - E_2).H < 0$ we have $h^0(2D_0 - H) = 0$, whence Φ_{H_D,ω_D} is surjective by Theorem 5.3(c).

Now by hypothesis $h^0(H-2D_0) = 0$ and we also have $h^0(2D_0-H+K_S) = h^0(E_3-E-E_2+K_S) = 0$, and by Riemann-Roch we get $h^1(H-2D_0) = 0$ as well. Therefore μ_{V_D,ω_D} is surjective by (14). Hence S is nonextendable by Proposition 5.1.

Case C: $h^0(E_2+E_3-E+K_S) = 0$**.** Set $D_0 = E+E_1+E_2+E_3$, which is nef (since $E+E_1+E_3$ is nef by Claim 11.5 and Lemma 6.4(b) and E_2 is nef by assumption) with $D_0^2 = 14$ and $\phi(D_0) = 3$. Moreover $H - D_0 \sim 2E + E_1$ is without fixed components.

We have $H-2D_0 \sim E-E_2-E_3$ and since $(H-2D_0).E = -2$ we have $h^0(E-E_2-E_3) = 0$. By hypothesis we have $h^2(E-E_2-E_3) = 0$, whence $h^1(H-2D_0) = 0$ by Riemann-Roch. It follows that μ_{V_D,ω_D} is surjective by (14).

Furthermore, since $2D_0-H \sim E_2+E_3-E$ and $h^0(E_2+E_3-E+K_S) = 0$ we have $h^0(2D_0-H) \leq 1$, and Φ_{H_D,ω_D} is surjective by Theorem 5.3(c)-(d). Hence S is nonextendable by Proposition 5.1. □

Lemma 11.9. *Suppose* $H \sim 3E + 2E_1 + E_2 + E_3$ *with* $E.E_1 = E.E_2 = E.E_3 = E_1.E_2 = E_1.E_3 = E_2.E_3 = 1$ *(the case* (31) *with* $\beta = 3$*). Then S is nonextendable.*

Proof. By Claim 11.5, Lemma 6.4(d) and symmetry, and adding K_S to both E_2 and E_3 if necessary, we can assume that $|E + E_2|$ is base-component free.

Now set $D_0 = 2E + 2E_1 + E_3$. Then $D_0^2 = 16$ and $\phi(D_0) = 3$. Hence Lemmas 6.2 and 6.4(b) give that D_0 is nef and $H - D_0 \sim E + E_2$ is base-component free.

We have $H - 2D_0 \sim -(2E_1 + E + E_3 - E_2)$ and we now prove that $h^0(2D_0 - H) = 2$ and $h^1(H - 2D_0) = 0$. To this end, by Theorem 4.13 and Riemann-Roch, we just need to show that

$B := 2E_1 + E + E_3 - E_2$ is quasi-nef. Let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.B \leq -2$. By Lemma 4.11 we can write $B \sim B_0 + k\Delta$ where $k = -\Delta.B \geq 2$, $B_0 > 0$ and $B_0^2 = B^2 = 2$. Now $2 = E.B = E.B_0 + kE.\Delta \geq 1 + 2E.\Delta$, therefore $E.\Delta = 0$. The ampleness of H implies that $E_2.\Delta \geq 2$, giving the contradiction $4 = E_2.B = E_2.B_0 + kE_2.\Delta \geq 5$. Therefore B is quasi-nef.

Now let $D \in |D_0|$ be a general curve. By [KL2, Cor.1] we know that $\operatorname{gon}(D) = 2\phi(D_0) = 6$ whence $\operatorname{Cliff}(D) = 4$, as D has genus 9 [ELMS, §5]. Therefore the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(e). Also μ_{V_D,ω_D} is surjective by (15) and S is nonextendable by Proposition 5.1. $\square$

11.3. **The case $M_2^2 = 4$.** We write $M_2 = E_2+E_3$ for primitive $E_2 > 0$ and $E_3 > 0$ with $E_2^2 = E_3^2 = 0$ and $E_2.E_3 = 2$, as in Lemma 10.1(b).

11.3.1. $\beta = 2$. By Lemma 10.1 we have $(E.E_2, E.E_3) = (1, 2)$ and the four cases $(E_1.E_2, E_1.E_3) = (1, 2)$, $(2, 1)$, $(2, 2)$ and $(1, 3)$. Note that in all cases $E_2.H < 2\phi(H) = 10$, whence E_2 is quasi-nef by Lemma 4.14.

If $(E_1.E_2, E_1.E_3) = (1, 2)$ we claim that either $E + E_2$ or $E_1 + E_2$ is nef. Indeed if there is a nodal curve Γ such that $\Gamma.(E + E_2) < 0$ then $\Gamma.E_2 = -1$ and $\Gamma.E = 0$. By Lemma 6.4(a) we have $\Gamma.E_1 > 0$, so that $E_2 \equiv E_1 + \Gamma$ and $E_1 + E_2 \equiv 2E_1 + \Gamma$ is nef. By symmetry the same arguments work if there is a nodal curve Γ such that $\Gamma.(E_1 + E_2) < 0$ and the claim is proved.

By symmetry between E and E_1 we can now assume that $E+E_2$ is nef. Setting $A := H-2E-2E_2$ we have $A^2 = 0$. As $E.A = 3$ and $E_2.A = 4$ we have that $A > 0$ is primitive and S is nonextendable by Lemmas 6.2 and 5.6(iii-b).

If $(E_1.E_2, E_1.E_3) = (2, 1)$ one easily sees that $H \sim 2(E_1 + E_2) + A$, with $A^2 = 0$, $E_1.A = 1$ and $E_2.A = 4$. Then $A > 0$ is primitive, $E_1 + E_2$ is nef by Lemma 6.4(e) and S is nonextendable by Lemmas 6.2 and 5.6(ii).

If $(E_1.E_2, E_1.E_3) = (1, 3)$ we have $(E_1 + E_3)^2 = 6$ and we can write $E_1 + E_3 \sim A_1 + A_2 + A_3$ with $A_i > 0$, $A_i^2 = 0$ and $A_i.A_j = 1$ for $i \neq j$. Then $E.A_i = E_1.A_i = E_2.A_i = E_3.A_i = 1$ and $A_i.H = 6$.

We now claim that either A_i is nef or $A_i \equiv E + \Gamma_i$ for a nodal curve Γ_i with $\Gamma_i.E = 1$. In particular, at least two of the A_i's are nef.

As a matter of fact if there is a nodal curve Γ with $\Gamma.A_i < 0$, then since $A_i.L_1 = 4 = \phi(L_1)$ we must have $\Gamma.L_1 \leq 0$, whence $\Gamma.E > 0$ by the ampleness of H and the first statement immediately follows. If two of the A_i's are not nef, say $A_1 \equiv E + \Gamma_1$ and $A_2 \equiv E + \Gamma_2$ then $1 = A_1.A_2 = (E + \Gamma_1).(E + \Gamma_2) = 2 + \Gamma_1.\Gamma_2$ yields the contradiction $\Gamma_1.\Gamma_2 = -1$ and the claim is proved.

We can therefore assume that A_1 and A_2 are nef. Let $A = H - 2A_1 - 2A_2$. Then $A^2 = 0$ and $E.A = 1$, whence $A > 0$ is primitive. As $A_1.A = A_2.A = 4$ and $\phi(H) = 5$, we have that S is nonextendable by Lemma 5.6(iii-b).

If $(E_1.E_2, E_1.E_3) = (2, 2)$, note first that E_1+E_2 is nef by Lemma 6.4(e). Set $A := H-2E_1-2E_2$. Then $A^2 = 0$ and $A.E = 1$, so that $A > 0$ is primitive. As $(E_1 + E_2).A = 6$, we have that S is nonextendable by Lemma 5.6(ii).

11.3.2. $\beta = 3$. By Lemma 10.1 we have $(E.E_2, E.E_3) = (1, 2)$ and $(E_1.E_2, E_1.E_3) = (1, 3)$ or $(2, 1)$.

We first show that E_i is quasi-nef for $i = 2, 3$. We have $H.E_2 \leq 9 < 2\phi(H) = 10$, whence E_2 is quasi-nef by Lemma 4.14. Now let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.E_3 \leq -2$. By Lemma 4.11 we can write $E_3 \sim A + k\Delta$, for $A > 0$ primitive with $A^2 = 0$, $k = -\Delta.E_3 = \Delta.A \geq 2$.

If $\Delta.E > 0$, from $E.E_3 = E.A + k\Delta.E$ we get that $k = 2$, $\Delta.E = 1$ and $E.A = 0$, whence the contradiction $E \equiv A$. Hence $\Delta.E = 0$.

We get the same contradiction if $\Delta.E_2 > 0$. Hence, by the ampleness of H we must have $\Delta.E_1 \geq 2$, but this gives the contradiction $E_1.E_3 = E_1.A + k\Delta.E_1 \geq 4$. Hence also E_3 is quasi-nef.

We now treat the case $(E_1.E_2, E_1.E_3) = (1, 3)$.

Let $D_0 = 2E + E_1 + E_2$. Then $D_0^2 = 10$ and $\phi(D_0) = 2$. Moreover D_0 and $H - D_0 \sim E + E_1 + E_3$ are base-point free by Lemma 6.4(a)-(b).

Moreover $2D_0 - H \sim E + E_2 - E_3$, and since $(2D_0 - H).E = -1$, we have $h^0(2D_0 - H) = 0$ and it follows from Theorem 5.3(c) that the map Φ_{H_D,ω_D} is surjective.

After possibly adding K_S to both E_2 and E_3, we can assume, by Lemmas 6.3 and 6.4(c), that the general members of both $|E + E_1|$ and $|E + E_2|$ are smooth irreducible curves. Let $D_1 \in |E + E_1|$ and $D_2 \in |E + E_2|$ be two such curves.

By Theorem 4.13 we have $h^1(H - D_0 - D_1) = h^1(E_3) = 0$, whence $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (16).

We now claim that $h^0(E_1 + E_3 - E_2) \leq 2$. Indeed, assume that $h^0(E_1 + E_3 - E_2) \geq 3$. Then $|E_1+E_3-E_2| = |M|+G$, with G the base-component and $|M|$ base-component free with $h^0(M) \geq 3$. If $M^2 = 0$, then $M \sim lP$, for an elliptic pencil P and an integer $l \geq 2$. But then $14 = (E_1 + E_3 - E_2).H = (lP + G).H \geq lP.H \geq 4\phi(H) = 20$, a contradiction. Hence $M^2 \geq 4$, but since $M.H \leq (E_1 + E_3 - E_2).H = 14$, this contradicts the Hodge index theorem.

Therefore we have shown that $h^0(E_1 + E_3 - E_2) \leq 2$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18). By Lemma 5.7, μ_{V_D,ω_D} is surjective and by Proposition 5.1, S is nonextendable.

Next we treat the case $(E_1.E_2.E_1.E_3) = (2, 1)$.

Let $D_0 = 2E + E_1 + E_3$. Then $D_0^2 = 14$, $\phi(D_0) = 3$ and D_0 and $H - D_0 \sim E + E_1 + E_2$ are base-point free by Lemma 6.4(a)-(b).

Moreover $2D_0-H \sim E+E_3-E_2$, and since $E+E_3$ is nef by Lemma 6.4(c) and $(2D_0-H).(E+E_3) = (E + E_3 - E_2).(E + E_3) = 1$, we get that $h^0(2D_0 - H) \leq 1$. It follows from Theorem 5.3(c)-(d) that the map Φ_{H_D,ω_D} is surjective.

Let $D_1 \in |E + E_1|$ and $D_2 \in |E + E_3|$ be two general members.

By Theorem 4.13 we have that $h^1(H-D_0-D_1) = h^1(E_2) = 0$, whence $\mu_{V_{D_1},\omega_{D_1}} = \mu_{\mathcal{O}_{D_1}(H-D_0),\omega_{D_1}}$. Since ω_{D_1} is a base-point free pencil we get that $\mu_{\mathcal{O}_{D_1(H-D_0)},\omega_{D_1}}$ is surjective by the base-point free pencil trick because $\deg(\mathcal{O}_{D_1}(H - D_0 - D_1 + K_S)) = 3$, whence $h^1(\mathcal{O}_{D_1}(H - D_0 - D_1 + K_S)) = 0$.

We have $(E_1 + E_2 - E_3).H = 5 = \phi(H)$, whence $h^0(E_1 + E_2 - E_3) \leq 1$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18).

By Lemma 5.7, μ_{V_D,ω_D} is surjective and, by Proposition 5.1, S is nonextendable.

11.4. **The case $M_2^2 = 6$.** By Lemma 10.1 we have $\beta = 2$ and $M_2 = E_2+E_3+E_4$ for primitive $E_i > 0$ with $E_i^2 = 0$, $E_i.E_j = 1$ for $i \neq j$ and $(E.E_2, E.E_3, E.E_4, E_1.E_2, E_1.E_3, E_1.E_4) = (1, 1, 2, 1, 1, 2)$. We note that E_1, E_2 and E_3 are nef by Lemma 4.14 and E_4 is quasi-nef by the same lemma.

By the ampleness of H it follows that $D_0 := E + E_1 + E_2 + E_3 + E_4$ is nef with $D_0^2 = 24$, $\phi(D_0) = 4$ and $H - D_0 \sim E + E_1$ is base-component free. Since $H - 2D_0 \sim -(E_2 + E_3 + E_4)$ we have $h^1(H - 2D_0) = 0$ by Theorem 4.13 and $h^0(2D_0 - H) = 4$ by Riemann-Roch. Then μ_{V_D,ω_D} is surjective by (15) and Φ_{H_D,ω_D} is surjective by Theorem 5.3(e), since $\mathrm{gon}(D) = 8$ by [KL2, Cor.1], whence Cliff $D = 6$ by [ELMS, §5], as $g(D) = 13$. Hence S is nonextendable by Proposition 5.1.

12. Case (II)

We have

$$H \equiv \beta E + \gamma E_1 + \delta E_2 + M_3, \quad E.E_1 = E.E_2 = E_1.E_2 = 1, \quad \beta, \gamma, \delta \in \{2, 3\},$$

$32 \leq H^2 \leq 52$ or $H^2 = 28$.

Since M_3 does not contain E, E_1 or E_2 in its arithmetic genus 1 decompositions, we have:

(36) $$\text{If } M_3 > 0 \text{ then } E.M_3 \geq \frac{1}{2}M_3^2 + 1,\ E_1.M_3 \geq \frac{1}{2}M_3^2 + 1 \text{ and } E_2.M_3 \geq \frac{1}{2}M_3^2 + 1.$$

Claim 12.1. *$E + E_1 + E_2$ is nef.*

Proof. Let Γ be a nodal curve such that $\Gamma.(E+E_1+E_2) < 0$. By Lemma 6.2 we must have $\Gamma.E_2 < 0$. We can then write $E_2 = A + k\Gamma$, for $A > 0$ primitive with $A^2 = 0$, $k = -\Gamma.E_2 = \Gamma.A \geq 1$.

Since $\phi(L_2) = E_2.L_2 = A.L_2 + k\Gamma.L_2 \leq A.L_2$, we must have $\Gamma.L_2 \leq 0$, whence either $\Gamma.E > 0$ or $\Gamma.E_1 > 0$, since H is ample. If $\Gamma.E > 0$, then $1 = E.E_2 = E.A + k\Gamma.E$, whence $k = 1$ and $A \equiv E$, which means $E_2 \equiv E + \Gamma$. But then $E_1.E_2 = 1$ yields $\Gamma.E_1 = 0$, whence $\Gamma.(E + E_1 + E_2) = 0$, a contradiction. We get the same contradiction if $\Gamma.E_1 > 0$. □

Set $B = E + E_1 + E_2$. Then $B^2 = 6$ and $(3B - H).B = 18 - 2(\beta + \gamma + \delta) - (E + E_1 + E_2).M_3$. If

$$2(\beta + \gamma + \delta) + (E + E_1 + E_2).M_3 \geq 17, \tag{37}$$

then $(3B - H).B \leq 1$, whence if $3B - H > 0$, it is a nodal cycle by Claim 12.1. Thus (37) implies that either $h^0(3B - H) = 0$ or $h^0(3B + K_S - H) = 0$ and S is nonextendable by Proposition 5.4.

We now deal with (37).

Assume first that $M_3 > 0$. Then, in view of (36), the condition (37) is satisfied unless $\beta = \gamma = \delta = 2$ and $(E + E_1 + E_2).M_3 = 3, 4$, which means that $M_3^2 = 0$, whence S is nonextendable by Claim 12.1 and Lemma 5.6(ii).

Assume now that $M_3 = 0$. Then the condition (37) is satisfied unless $6 \leq \beta + \gamma + \delta \leq 8$. Since $E.H = \gamma + \delta$ and $E_1.H = \beta + \delta$, we get $\gamma \leq \beta$. At the same time, since $E_1.L_1 = \beta - \alpha + \delta$ and $E_2.L_1 = \beta - \alpha + \gamma$, we get $\gamma \geq \delta$. Recalling that we assume that H is not 2-divisible in $\operatorname{Num} S$, we end up with the cases $(\beta, \gamma, \delta) = (3, 2, 2)$ or $(3, 3, 2)$.

The first case has $g = 17$ and is case (a3) in the proof of Proposition 15.1. In the second case, set $D_0 = 2E + E_1 + E_2$, which is nef by Claim 12.1 and satisfies $D_0^2 = 10$ and $\phi(D_0) = 2$. Note that E_1 is nef by Lemma 4.14 since $E_1.H = E.H = \phi(H)$. Now $H - D_0 \equiv E + 2E_1 + E_2$ is nef by Claim 12.1 with $(H - D_0)^2 = 10$ and $\phi(H - D_0) = 2$, whence $|H - D_0|$ is base-point free. We have $H - 2D_0 \equiv E_1 - E$, so that $(H - 2D_0)^2 = -2$ and $(H - 2D_0).H = 0$. Therefore, by Riemann-Roch, $h^i(H - 2D_0) = h^i(H - 2D_0 + K_S) = 0$ for all $i = 0, 1, 2$. By Theorem 5.3(c) Φ_{H_D,ω_D} is surjective. Moreover μ_{V_D,ω_D} is surjective by (14). Therefore S is nonextendable by Proposition 5.1.

13. Case (III)

We have

$$H \equiv \beta E + \gamma E_1 + M_2, \quad E.E_1 = 2, \quad \beta, \gamma \in \{2, 3\},$$

$32 \leq H^2 \leq 62$ or $H^2 = 28$ and L_2 is of small type. In particular

$$\phi(H) = E.H = 2\gamma + E.M_2 \leq 7 \tag{38}$$

and

$$\text{either } M_2 > 0 \text{ or } \beta = \gamma = 3. \tag{39}$$

Since M_2 contains neither E nor E_1 in its arithmetic genus 1 decompositions, we have:

$$\text{If } M_2 > 0 \text{ then } E.M_2 \geq \frac{1}{2}M_2^2 + 1 \text{ and } E_1.M_2 \geq \frac{1}{2}M_2^2 + 1. \tag{40}$$

By Proposition 5.5 and Lemma 6.2 we have that S is nonextendable if $(E + E_1).H \geq 17$, therefore in the following we can assume

$$(E + E_1).H = 2(\beta + \gamma) + (E + E_1).M_2 \leq 16. \tag{41}$$

We now divide the rest of the treatment into the cases $\beta = 2$ and $\beta = 3$.

13.1. **The case $\beta = 2$.** We have $M_2 > 0$ by (39) and $E.M_2 \geq 1$ by (40).

If $\gamma = 3$, then $E.M_2 = 1$ and $\phi(H) = 7$ by (38), so that $M_2^2 = 0$ by (40). As $L_2 \equiv E_1 + M_2$ the removing conventions of Section 6, page 19, require $E_1.L_2 < E.L_2$. Hence $E_1.M_2 \leq 2$, giving the contradiction $49 = \phi(H)^2 \leq H^2 \leq 40$.

Therefore $\gamma = 2$, so that $E.M_2 \leq 3$ by (38), whence $M_2^2 \leq 4$ by (40). Moreover $(E + E_1).M_2 \leq 8$ by (41), whence

$$\phi(H)^2 = (4 + E.M_2)^2 \leq H^2 = 16 + M_2^2 + 4(E + E_1).M_2 \leq 48 + M_2^2. \tag{42}$$

Combining with [KL2, Prop.1] we get $E.M_2 \leq 2$, whence $M_2^2 \leq 2$ by (40).

We now treat the two cases $M_2^2 = 0$ and $M_2^2 = 2$ separately.

If $M_2^2 = 2$, then $E.M_2 = 2$ by (40) and since $(E_1.M_2)^2 = \phi(L_1)^2 \leq L_1^2 = 4E_1.M_2 + 2$, we must have $E_1.M_2 \leq 4$. Writing $M_2 \sim E_2 + E_3$ for isotropic $E_2 > 0$ and $E_3 > 0$ with $E_2.E_3 = 1$, we must have $E.E_2 = E.E_3 = 1$. As $E_i.H \geq \phi(H) = E.H = 6$ for $i = 2, 3$, we find the only possibility $E_1.E_2 = E_1.E_3 = 2$.

Since $H.E_2 = H.E_3 = 7 < 2\phi(H)$ we have that E_2 and E_3 are quasi-nef by Lemma 4.14, whence $E + E_1 + E_2$ and $E + E_1 + E_3$ are nef by Lemma 6.4(b).

Set $D_0 = E + E_1 + E_2$. Then $D_0^2 = 10$, $\phi(D_0) = 3$ and both D_0 and $H - D_0$ are base-point free.

Now $H - 2D_0 \equiv E_3 - E_2$ and $(H - 2D_0)^2 = -2$ with $(H - 2D_0).H = 0$. Therefore $h^i(2D_0 - H) = h^i(2D_0 - H + K_S) = 0$ for $i = 0, 1, 2$. By Theorem 5.3(c) Φ_{H_D,ω_D} is surjective. Moreover μ_{V_D,ω_D} is surjective by (14). By Proposition 5.1 we get that S is nonextendable.

Finally, if $M_2^2 = 0$, then S is nonextendable by Lemmas 6.2 and 5.6(ii) unless $(E + E_1).M_2 \leq 3$. In the latter case, by (42), we get $E.M_2 = 1$, whence $M_2^2 = 0$ by (40) and $E_1.M_2 = 2$.

Set $E_2 := M_2$ and grant for the moment the following:

Claim 13.1. *There is an isotropic effective* 10*-sequence* $\{f_1, \ldots, f_{10}\}$*, with* $f_1 = E$*,* $f_{10} = E_2$*, all* f_i *nef for* $i \leq 9$*, and, for each* $i = 1, \ldots, 9$*, there is an effective decomposition* $H \sim 2f_i + 2g_i + h_i$*, where* $g_i > 0$ *and* $h_i > 0$ *are primitive, isotropic with* $f_i.g_i = g_i.h_i = 2$ *and* $f_i.h_i = 1$*. Furthermore,* $g_i + h_i$ *is not nef for at most one* $i \in \{1, \ldots, 9\}$*.*

By the claim we can assume that $H \sim 2E + 2E_1 + E_2$ with $E_1 + E_2$ nef. We have $(E_1 + E_2 - E)^2 = -2$. Since $1 = (E_1 + E_2).(E_1 + E_2 - E) < \phi(E_1 + E_2) = 2$ we have that if $E_1 + E_2 - E > 0$ it is a nodal cycle, whence either $h^0(E_1 + E_2 - E) = 0$ or $h^0(E_1 + E_2 - E + K_S) = 0$. By replacing E with $E + K_S$ if necessary, we can assume that $h^0(E_1 + E_2 - E) = 0$. As $h^2(E_1 + E_2 - E) = h^0(E - E_1 - E_2 + K_S) = 0$ by the nefness of E, we find from Riemann-Roch that $h^1(E_1 + E_2 - E) = 0$ as well.

Set $D_0 = 2E + E_1$, so that D_0 is nef by Lemma 6.2 with $D_0^2 = 8$ and $\phi(D_0) = 2$. Moreover $H - D_0 = E_1 + E_2$ is nef by assumption, with $\phi(H - D_0) = 2$, whence base-point free.

We have $2D_0 - H = 2E - E_2$, and since $(2D_0 - H).E = -1$, we have $h^0(2D_0 - H) = 0$ and by Theorem 5.3(c) we get that Φ_{H_D,ω_D} is surjective.

Now let $D_1 = E$ and $D_2 \in |E + E_1|$ be a general smooth irreducible curve. Since $h^1(H - D_0 - D_1) = h^1(E_1 + E_2 - E) = 0$, we have that $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (17).

Now $h^0(H - D_0 - D_2) = h^0(E_2 - E) \leq h^0(E_1 + E_2 - E) = 0$, whence $h^0(E_2 - 2E) = h^0(E_2 - E) = 0$, so that $h^1(E_2 - 2E) = 1$ by Riemann-Roch. Therefore $h^0(\mathcal{O}_{D_2}(H - D_0 - D_1)) = h^0(\mathcal{O}_{D_2}(E_1 + E_2 - E)) \leq h^0(E_1 + E_2 - E) + h^1(E_2 - 2E) = 1$ and $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18). By Lemma 5.7, μ_{V_D,ω_D} is surjective and by Proposition 5.1, S is nonextendable.

We have left to prove the claim.

Proof of Claim 13.1. Let $Q = E + E_1 + E_2$. Then $Q^2 = 10$ and $\phi(Q) = 3$, therefore, by [CD, Cor.2.5.5], there is an isotropic effective 10-sequence $\{f_1, \ldots, f_{10}\}$ such that $3Q \sim f_1 + \ldots + f_{10}$. Since $E.Q = E_2.Q = 3$ we can without loss of generality assume that $f_1 = E$ and $f_{10} = E_2$. Now

$Q \sim f_1 + f_{10} + E_1$, whence $f_i.E_1 = 1$ for all $i \in \{2, \dots, 9\}$. It follows that $f_i.H = \phi(H) = 5$ for all $i \le 9$, whence all f_i are nef for $i \le 9$ by Lemma 4.14.

Now for $i \le 9$ we have $(H-2f_i)^2 = 8$. If $\phi(H-2f_i) = 1$, then $H-2f_i = 4F_1+F_2$ for $F_i > 0$, $F_i^2 = 0$ and $F_1.F_2 = 1$, but then $f_i.H = 5$ implies $f_i.F_1 = 1$, so that $F_1.H = 3$, a contradiction. Therefore $\phi(H-2f_i) = 2$, so that $H-2f_i = 2g_i + h_i$ for isotropic $g_i > 0$ and $h_i > 0$ with $g_i.h_i = 2$. Moreover g_i is primitive since it computes $\phi(H-2f_i)$. Now $5 \le g_i.H = 2+2f_i.g_i$ implies $f_i.g_i \ge 2$, and $f_i.H = 5$ implies $f_i.g_i = 2$ and $f_i.h_i = 1$, so that h_i is primitive. Moreover $H.g_i = H.h_i = 6 < 2\phi(H)$, whence g_i and h_i are quasi-nef by Lemma 4.14.

Assume that $g_i + h_i$ is not nef for some $i \le 9$ and let R be a nodal curve with $R.(g_i + h_i) < 0$.

If $R.g_i < 0$ then $R.g_i = -1$ and $R.(2g_i + h_i) \le -2$, whence $R.f_i \ge 2$ by the ampleness of H. By Lemma 4.11 we can write $g_i \sim A + R$, with $A > 0$ primitive such that $A^2 = 0$ and $A.R = 1$. From $2 = f_i.g_i = f_i.A + f_i.R$ we get $f_i.R = 2$ and $f_i \equiv A$, a contradiction.

Therefore $R.g_i = 0$, $R.h_i = -1$ and as above we can write $h_i \sim A + R$, with $A > 0$ primitive such that $A^2 = 0$ and $A.R = 1$. Now $R.f_i > 0$ by the ampleness of H, and again by $1 = f_i.h_i = f_i.A + f_i.R$ we get $f_i.R = 1$ and $f_i \equiv A$, so that $h_i \equiv f_i + R$ with $R.f_i = 1$.

It follows that if $g_i + h_i$ and $g_j + h_j$ are not nef for two distinct $i, j \le 9$, say for $i = 1$ and $j = 2$ for simplicity, then $h_1 \equiv f_1 + R_1$ and $h_2 \equiv f_2 + R_2$ where R_1 and R_2 are nodal curves such that $R_1.f_1 = R_2.f_2 = 1$. Then

$$H \equiv 3f_1 + 2g_1 + R_1 \equiv 3f_2 + 2g_2 + R_2.$$

Now the nefness of f_2 and

$$5 = f_2.H = 3 + 2g_1.f_2 + R_1.f_2$$

imply that $g_1.f_2 = 1$ and $R_1.f_2 = 0$. As $(R_1 + R_2).H = 2 < \phi(H)$, we get $R_1.R_2 \le 1$. Hence

$$1 = R_1.H = 3R_1.f_2 + 2R_1.g_2 + R_1.R_2 = 2R_1.g_2 + R_1.R_2,$$

so that $R_1.g_2 = 0$ and $R_1.R_2 = 1$. Similarly $R_2.g_1 = 0$, whence we get the absurdity

$$6 = g_1.H = 3g_1.f_2 + 2g_1.g_2 + g_1.R_2 = 3 + 2g_1.g_2.$$

Therefore $g_i + h_i$ is not nef for at most one $i \le 9$ and the claim is proved. □

13.2. **The case $\beta = 3$.** Replacing E with $E + K_S$ if necessary, we can assume that $H \sim 3E + \gamma E_1 + M_2$. We first claim that

$$(\gamma - 1)E_1 + M_2 \text{ and } (\gamma - 2)E_1 + M_2 \text{ are quasi-nef.} \tag{43}$$

Let $\varepsilon = 0, 1$ and let $\Delta > 0$ be such that $\Delta^2 = -2$ and $\Delta.((\gamma - 1 - \varepsilon)E_1 + M_2) \le -2$.

If $\Delta.E_1 < 0$, then $\Delta.E \ge 2$ by the ampleness of H. By Lemma 4.11 we can write $E_1 \sim A + k\Delta$ with $A > 0$ primitive, $A^2 = 0$ and $k = -E_1.\Delta = A.\Delta \ge 1$. But then $2 = E.E_1 = E.A + kE.\Delta$ implies the contradiction $k = 1$, $E.\Delta = 2$ and $E \equiv A$.

Hence $\Delta.E_1 \ge 0$, so that $M_2 > 0$ and $l := -\Delta.M_2 \ge 2$. By Lemma 4.11 we can write $M_2 \sim A_2 + l\Delta$ with $A_2 > 0$ primitive, $A_2^2 = M_2^2$ and $\Delta.A_2 = l$.

If $\Delta.E = 0$, then $\Delta.E_1 \ge 2$ by ampleness of H, whence $E_1.M_2 = E_1.(A_2 + l\Delta) \ge 4$, so that $\gamma = 2$ by (41), which moreover implies $E_1.M_2 \le 5$, so that $l = E_1.\Delta = 2$. As $(E_1 + \Delta)^2 = 2$, we must have

$$2\phi(L_1) \le (E_1 + \Delta).L_1 = \phi(L_1) + \Delta.((3 - \alpha)E + 2E_1 + M_2) = \phi(L_1) + 2,$$

and we get the contradiction $4 \le E_1.M_2 \le E_1.L_1 = \phi(L_1) \le 2$.

Therefore $\Delta.E > 0$, so that $E.M_2 = E.(A_2 + l\Delta) \ge 3$, whence $E.M_2 = 3$, $\gamma = 2$ and $\phi(H) = 7$ by (38), whence $M_2^2 \le 4$ by (40). By (41) we must have $E_1.M_2 \le 3$, but as $H^2 = 42+4E_1.M_2+M_2^2 \ge 54$ by [KL2, Prop.1], using (40), we must have $E_1.M_2 = 3$. Since $E_1.(H - 2E) = 5 \le \phi(H) = 7$ we have $\alpha = 2$, $L_1 \sim E + 2E_1 + M_2$ and $L_2 \sim E + M_2$. Since the latter is of small type and $M_2^2 \le 4$, we must have either $M_2^2 = 0$ or $M_2^2 = 4$. In the latter case we get $L_2^2 = 10$ and $\phi(L_2) = 3$. Now $(E + \Delta)^2 \ge 0$ and $(E + \Delta).M_2 \le 1$, whence $\phi(M_2) = 1$ and we can write $M_2 \sim 2F_1 + F_2$ for some $F_i > 0$ with

$F_i^2 = 0$ and $F_1.F_2 = 1$. Therefore $3 = \phi(L_2) \leq F_1.L_2 = F_1.E + 1$, so that $F_1.E \geq 2$, giving the contradiction $3 = E.M_2 = 2F_1.E + F_2.E \geq 4$. Hence $M_2^2 = 0$, $L_1^2 = 26$ and $\phi(L_1) = E_1.L_1 = 5$, contradicting [KL2, Prop.1]. Therefore (43) is proved.

To show that S is nonextendable set $D_0 = 2E + E_1$, which is nef by Lemma 6.2 with $D_0^2 = 8$ and $\phi(D_0) = 2$. Moreover $H - D_0 \sim E + (\gamma - 1)E_1 + M_2$ is easily seen to be nef by (43). Since $\phi(H - D_0) \geq 2$ we have that $H - D_0$ is base-point free.

We have $h^0(2D_0 - H) = h^0(E + (2 - \gamma)E_1 - M_2) = 0$ by the nefness of E and (39), whence the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(c).

If $M_2 > 0$ and $(\gamma, E.M_2, E_1.M_2) = (2, 1, 1)$, then $M_2^2 = 0$ by (40), $(H - 2D_0)^2 = -2$ and $(H - 2D_0).H = 0$, whence $h^1(H - 2D_0) = 0$ by Riemann-Roch, so that μ_{V_D,ω_D} is surjective by (14), as E_1 is primitive.

In the remaining cases, to show the surjectivity of μ_{V_D,ω_D} we apply Lemma 5.7 with $D_1 = E + K_S$ and D_2 general in $|E + E_1 + K_S|$.

Since $h^1(H - D_0 - D_1) = h^1((\gamma - 1)E_1 + M_2 + K_S) = 0$ by (43) and Theorem 4.13, we have that $\mu_{V_{D_1},\omega_{D_1}}$ is surjective by (17).

From (43) and Theorem 4.13 we also have $h^1(H - D_0 - D_2) = h^1((\gamma - 2)E_1 + M_2 + K_S) = 0$, whence $\mu_{V_{D_2},\omega_{D_2}(D_1)} = \mu_{\mathcal{O}_{D_2}(H-D_0),\omega_{D_2}(D_1)}$. This is surjective by [Gr, Cor.4.e.4] if $M_2 > 0$, since we assume $(\gamma, E.M_2, E_1.M_2) \neq (2, 1, 1)$.

Finally, if $M_2 = 0$, then $\gamma = 3$ by (39), whence $H.E_1 = 6 = \phi(H)$, so that E_1 is nef by Lemma 4.14 and $h^0(H - D_0 - D_2) = h^0(E_1 + K_S) = 1$. We get $h^0(\mathcal{O}_{D_2}(H - D_0 - D_1)) \leq h^0(H - D_0 - D_1) + h^1(H - 2D_0) = 2$, since $h^0(H - D_0 - D_1) = h^0(2E_1 + K_S) = 1$ and $h^1(H - 2D_0) = h^1(E_1 - E) = 1$ by Riemann-Roch and the nefness of E and E_1. Hence $\mu_{V_{D_2},\omega_{D_2}(D_1)}$ is surjective by (18).

Therefore μ_{V_D,ω_D} is surjective in all cases and S is nonextendable by Proposition 5.1.

14. Case (S)

We have $H \sim \alpha E + L_1$ with $L_1^2 > 0$ by Lemma 4.10 and L_1 of small type by hypothesis. Also we assume that H is not numerically 2-divisible in $\operatorname{Num} S$ and $H^2 \geq 32$ or $H^2 = 28$.

If $\alpha = 2$ we get $H^2 = 4E.L_1 + L_1^2 = 4\phi(H) + L_1^2$, whence $(\phi(H))^2 \leq 4\phi(H) + L_1^2$ and Lemma 4.8 yield $\phi(H) \leq 5$, incompatible with the hypotheses on H^2. Therefore $\alpha \geq 3$ and we can write $L_1 \sim F_1 + \ldots + F_k$ as in Lemma 4.8 with $k = 2$ or 3 and $E.F_1 \geq \ldots \geq E.F_k$.

If $E.F_k = 0$ then $E \equiv F_k$ and $3 \leq \phi(H) = E.H = E.L_1 = F_k.L_1$, so that $E.L_1 = F_k.L_1 = 3$, $L_1^2 = 10$ and we can write $L_1 \sim E + E_1 + E_2$ with $(E.E_1, E.E_2, E_1.E_2) = (1, 2, 2)$.

If $E.F_k > 0$, by definition of α we must have

$$\phi(H) + 1 \leq F_k.(L_1 + E) \leq F_k.L_1 + \frac{1}{k}E.L_1 = F_k.L_1 + \frac{1}{k}\phi(H),$$

whence $F_k.L_1 = 3$ or 4, $L_1^2 = 10$, $k = 3$ and $\phi(H) = 3$ or 4. Hence we can decompose $L_1 \sim E + E_1 + E_2$ with $(E.E_1, E.E_2, E_1.E_2) = (1, 2, 2)$ if $\phi(H) = 3$ and $(E.E_1, E.E_2, E_1.E_2) = (2, 2, 1)$ if $\phi(H) = 4$. Therefore, setting $\beta = \alpha + 1$, we get the following cases:

$$H \sim \beta E + E_1 + E_2, \quad \beta \geq 4, \quad E.E_1 = 1, E.E_2 = E_1.E_2 = 2, \tag{44}$$

$$H \sim \beta E + E_1 + E_2, \quad \beta \geq 4, \quad E.E_1 = E.E_2 = 2, E_1.E_2 = 1. \tag{45}$$

Claim 14.1. (i) *In the cases* (44) *and* (45) *we have that $E + E_2$ is nef and E_2 is quasi-nef.*

(ii) *In case* (44) *both $nE + E_2 - E_1$ and $nE + E_2 - E_1 + K_S$ are effective and quasi-nef for all $n \geq 2$, and moreover they are primitive and isotropic for $n = 2$.*

Proof. Assume that $\Delta > 0$ satisfies $\Delta^2 = -2$ and $\Delta.E_2 = -k$ for some $k > 0$. By Lemma 4.11 we can write $E_2 = A + k\Delta$, for $A > 0$ primitive with $A^2 = 0$ and $A.\Delta = k$. If $\Delta.E = 0$ the ampleness of H yields $\Delta.E_1 \geq 2$, and, from $E_1.E_2 = E_1.A + kE_1.\Delta$, we get $E_1.E_2 = 2$, $k = 1$ and $E_1.A = 0$,

whence the contradiction $E_1 \equiv A$. Therefore $\Delta.E > 0$ and it follows that if $\Delta.(E+E_2) < 0$, then $\Delta.E_2 \leq -2$. Hence we can assume $k \geq 2$ and we get from $2 = E.E_2 = E.A + kE.\Delta$ that $k = 2$, $E.\Delta = 1$ and $E.A = 0$, whence the contradiction $E \equiv A$. This proves (i).

As for (ii), note that $(2E+E_2-E_1)^2 = 0$ and $(E+E_2).(2E+E_2-E_1) = 3 < 2\phi(E+E_2) = 4$, so that $h^0(2E+E_2-E_1) = h^0(2E+E_2-E_1+K_S) = 1$ by Lemma 4.14, whence also $h^1(2E+E_2-E_1) = h^1(2E+E_2-E_1+K_S) = 0$ by Riemann-Roch. Since $E.(2E+E_2-E_1) = 1$, the statement follows for $n = 2$ by Theorem 4.13, and consequently for all $n \geq 2$ again by the same theorem. □

Lemma 14.2. *Let H be as in (44) or (45). Then S is nonextendable.*

Proof. We first treat the case (44) with $\beta = 4$.

In this case we set $D_0 = 3E + E_2$, which is nef by Claim 14.1(i) with $D_0^2 = 12$. Then $H - D_0 \sim E + E_1$ is a base-component free pencil by Lemma 6.3 and $H - 2D_0 \sim -2E + E_1 - E_2$. By Claim 14.1(ii) we have $h^0(2D_0 - H) = 1$, so that the map Φ_{H_D,ω_D} is surjective by Theorem 5.3(d), and $h^1(H - 2D_0) = 0$ so that μ_{V_D,ω_D} is surjective by (14). By Proposition 5.1 we find that S is nonextendable.

In the general case we set $D_0 = kE + E_2$ with $k = \lfloor \frac{\beta}{2} \rfloor \geq 2$. Then $D_0^2 = 4k \geq 8$ and D_0 is nef by Claim 14.1(i) with $\phi(D_0) = 2$. We have $H - D_0 \sim (\beta - k)E + E_1$, whence by Lemma 6.3 we deduce that $H - D_0$ is base-component free.

Since $2D_0 - H \sim (2k-\beta)E + E_2 - E_1 \leq E_2 - E_1$ we have $h^0(2D_0 - H) = 0$ as $(E+E_2).(E_2-E_1) = -1$ in case (44) and $H.(E_2 - E_1) = 0$ in (45). Hence Φ_{H_D,ω_D} is surjective by Theorem 5.3(c).

Now if β is even and we are in case (45) we have $h^0(H-2D_0) = h^2(H-2D_0) = 0$ as $H.(H-2D_0) = H.(E_2 - E_1) = 0$. It follows that $h^1(H - 2D_0) = 0$ and consequently μ_{V_D,ω_D} is surjective by (14) since E_2 is primitive. Hence S is nonextendable by Proposition 5.1.

We can therefore assume that β is odd in case (45). In particular $\beta \geq 5$ for the rest of the proof and, by Proposition 5.1, we just need to prove the surjectivity of μ_{V_D,ω_D}, for which we will use Lemma 5.8.

We have $D_0 + K_S - 2E \sim (k-2)E + E_2 + K_S$, whence $h^1(D_0 + K_S - 2E) = 0$ by Claim 14.1(i) and Theorem 4.13. Moreover $h^2(D_0 + K_S - 4E) = h^0((4-k)E - E_2) = 0$ by the nefness of E.

Since $H - D_0 - 2E \sim (\beta-k-2)E + E_1$ and $\beta - k - 2 \geq 1$ as $\beta \geq 5$, we have that $|H - D_0 - 2E|$ is base-component free by Lemma 6.3. Since $(E+E_2).(-E+E_1-E_2) < 0$ we have that $h^0(H-2D_0-2E) = h^0((\beta-2k-2)E + E_1 - E_2) \leq h^0(-E + E_1 - E_2) = 0$, whence (11) is equivalent to

$$h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq (\beta - k - 2)E.E_1 - 1. \tag{46}$$

In the case (45) with $\beta = 5$ we have that $\deg \mathcal{O}_D(H - D_0 - 4E) = (-E + E_1).(2E + E_2) = 3$ and D is nontrigonal by [KL2, Cor.1], whence $h^0(\mathcal{O}_D(H - D_0 - 4E)) \leq 1$ and (46) is satisfied.

Hence we can assume, for the rest of the proof, that $\beta \geq 5$ in case (44) and $\beta \geq 7$ (and odd) in case (45). This implies $\beta - k - 4 \geq -1$ in case (44) and ≥ 0 in case (45), so that we have $h^0((\beta - k - 4)E + E_1) = (\beta - k - 4)E.E_1 + 1$ by Lemma 6.3 and Riemann-Roch. Hence

$$\begin{aligned} h^0(\mathcal{O}_D(H - D_0 - 4E)) &\leq h^0(H - D_0 - 4E) + h^1(H - 2D_0 - 4E)) \leq \\ &\leq (\beta - k - 4)E.E_1 + 1 + h^1(K_S + (2k + 4 - \beta)E + E_2 - E_1), \end{aligned}$$

and to prove (46) it remains to show

$$h^1(K_S + (2k + 4 - \beta)E + E_2 - E_1) \leq 2E.E_1 - 2. \tag{47}$$

In case (44) we have $2k + 4 - \beta = 3$ or 4, and (47) follows from Claim 14.1(ii).

In case (45) we have $2k + 4 - \beta = 3$, and as $(3E + E_2 - E_1)^2 = -2$ and $h^2(K_S + 3E + E_2 - E_1) = h^0(E_1 - 3E - E_2) = 0$, we have that (47) is equivalent to $h^0(K_S + 3E - E_1 + E_2) \leq 2$. If, by contradiction, $h^0(K_S + 3E - E_1 + E_2) \geq 3$, then we can write $|K_S + 3E - E_1 + E_2| = |M| + \Delta$ for Δ fixed and $h^0(M) \geq 3$. Since $E.(K_S + 3E - E_1 + E_2) = 0$ and E is nef, we must have

$E.M = E.\Delta = 0$, whence $M \sim 2lE$ for an integer $l \geq 2$ and $E_2.\Delta \geq 0$ by the nefness of $E + E_2$. Now $5 = E_2.(K_S + 3E - E_1 + E_2) \geq 4l \geq 8$, a contradiction. Hence (47) is proved. □

15. Main theorem and surfaces of genus 15 and 17

We have shown, throughout Sections 7-14, that every Enriques surface $S \subset \mathbb{P}^r$ of genus $g \geq 18$ is nonextendable, thus proving our main theorem.

Moreover the theorem can be made more precise in the cases $g = 15$ and $g = 17$:

Proposition 15.1. *Let $S \subset \mathbb{P}^r$ be a smooth Enriques surface, let H be its hyperplane bundle let $E > 0$ such that $E.H = \phi(H)$ and suppose that either $H^2 = 32$ or $H^2 = 28$. Then S is nonextendable if H satisfies:*

(a) *$H^2 = 32$ and either $\phi(H) \neq 4$ or $\phi(H) = 4$ and neither H nor $H - E$ are 2-divisible in $\operatorname{Pic} S$.*
(b) *$H^2 = 28$ and either $\phi(H) = 5$ or $(\phi(H), \phi(H - 3E)) = (4, 2)$ or $(\phi(H), \phi(H - 4E)) = (3, 2)$.*

Proof. We have shown, throughout Sections 7-14, that S is nonextendable unless it has one of the following ladder decompositions:

(a1) $H \sim 4E + 4E_1$, $E.E_1 = 1$, $H^2 = 32$ (page 22);
(a2) $H \sim 4E + 2E_1$, $E.E_1 = 2$, $H^2 = 32$ (page 22);
(a3) $H \sim 3E + 2E_1 + 2E_2$, $E.E_1 = E.E_2 = E_1.E_2 = 1$, $H^2 = 32$ (page 38).
(b1) $H \sim 3E + 2E_1 + E_2$, $E.E_1 = E_1.E_2 = 1$, $E.E_2 = 2$, $H^2 = 28$ (page 30);
(b2) $H \sim 4E + 2E_1 + E_2$, $E.E_1 = E.E_2 = E_1.E_2 = 1$, $H^2 = 28$ (page 32).

Now in the cases (a1)-(a3) we have $\phi(H) = 4$ and we see that H is 2-divisible in $\operatorname{Pic} S$ in the cases (a1) and (a2) and $H - E$ is 2-divisible in $\operatorname{Pic} S$ in case (a3). In case (b1) we have $(\phi(H), \phi(H - 3E)) = (4, 1)$ whereas in case (b2) we have $(\phi(H), \phi(H - 4E)) = (3, 1)$. □

16. A new Enriques-Fano threefold

We know by the articles of Bayle [Ba, Thm.A] and Sano [Sa, Thm.1.1] that for every g such that $6 \leq g \leq 10$ or $g = 13$ there is an Enriques-Fano threefold in $\mathbb{P}^g$. As mentioned in the introduction there has been some belief that the examples found by Bayle and Sano exhaust the complete list of Enriques-Fano threefolds. We will see in this section that this is not so (see also [P2, Prop.3.2 and Rmk.3.3]).

We now prove a more precise version of Proposition 1.4.

Proposition 16.1. *There exists an Enriques-Fano threefold $X \subseteq \mathbb{P}^9$ of genus 9 with the following properties:*

(a) *X does not have a $\mathbb{Q}$-smoothing. In particular, it does not lie in the closure of the component of the Hilbert scheme made of Fano-Conte-Murre-Bayle-Sano's examples.*
(b) *Let $\mu : \widetilde{X} \to X$ be the normalization. Then $\widetilde{X}$ has canonical but not terminal singularities, it does not have a $\mathbb{Q}$-smoothing and $(\widetilde{X}, \mu^* \mathcal{O}_X(1))$ does not belong to the list of Fano-Conte-Murre-Bayle-Sano.*
(c) *On the general smooth Enriques surface $S \in |\mathcal{O}_X(1)|$, we have $\mathcal{O}_S(1) \cong \mathcal{O}_S(2E + 2E_1 + E_2)$, where E, E_1 and E_2 are smooth irreducible elliptic curves with $E.E_1 = E.E_2 = E_1.E_2 = 1$.*

Proof. Let $\overline{X} \subset \mathbb{P}^{13}$ be the well-known Enriques-Fano threefold of genus 13. By [Fa, CM] we have that $\overline{X} \subset \mathbb{P}^{13}$ is the image of the blow-up of $\mathbb{P}^3$ along the edges of a tetrahedron, via the linear system of sextics double along the edges.

This description of $\overline{X}$ allows to identify the linear system embedding the general hyperplane section $\overline{S} = \overline{X} \cap \overline{H} \subset \mathbb{P}^{12}$. Let $P_1, \ldots, P_4$ be four independent points in $\mathbb{P}^3$, let l_{ij} be the line joining P_i and P_j and denote by $\widetilde{\mathbb{P}}^3$ the blow-up of $\mathbb{P}^3$ along the l_{ij}'s with exceptional divisors

E_{ij} and by $\widetilde{H}$ the pull-back of a plane in $\mathbb{P}^3$. Let $\widetilde{L} = 6\widetilde{H} - 2\sum_{1 \leq i < j \leq 4} E_{ij}$. Therefore $\overline{S}$ is just a general element $\widetilde{S} \in |\widetilde{L}|$, embedded with $\widetilde{L}_{|\widetilde{S}}$. Now let $\widetilde{l}_{ij}$ be the inverse image of l_{ij} on $\widetilde{S}$. Then by [GH, Ch.4, §6, page 634], for each pair of disjoint lines l_{ij}, l_{kl} on $\widetilde{S}$ there is a genus one pencil $|2\widetilde{H}_{|\widetilde{S}} - \widetilde{l}_{ik} - \widetilde{l}_{il} - \widetilde{l}_{jk} - \widetilde{l}_{jl}| = |2\widetilde{l}_{ij}|$. Therefore $\widetilde{L}_{|\widetilde{S}} \sim 2\widetilde{l}_{12} + 2\widetilde{l}_{13} + 2\widetilde{l}_{14}$ and we have decomposed the hyperplane bundle of $\overline{S} \subset \mathbb{P}^{12}$ as $2E + 2E_1 + 2E_2$ where $E := \widetilde{l}_{12}, E_1 := \widetilde{l}_{13}, E_2 := \widetilde{l}_{14}$ are half-pencils and $E.E_1 = E.E_2 = E_1.E_2 = 1$. Also E, E_1 and E_2 are smooth and irreducible.

To find a new Enriques-Fano threefold X of genus 9 we consider the linear span $M \cong \mathbb{P}^3$ of E_2, the projection $\pi_M : \mathbb{P}^{13} - - \to \mathbb{P}^9$ and let $X = \pi_M(\overline{X}) \subset \mathbb{P}^9$.

Let $\psi : \widetilde{\overline{X}} \to \overline{X}$ be the blow up of $\overline{X}$ along E_2 with exceptional divisor F and set $\mathcal{H} = (\psi^* \mathcal{O}_{\overline{X}}(1))(-F)$ and let $\widetilde{\overline{S}} \in |\mathcal{H}| \cong |\mathcal{I}_{E_2/\overline{X}}(1)|$ be the smooth Enriques surface isomorphic to $\overline{S}$.

Then, by

$$0 \longrightarrow \mathcal{O}_{\widetilde{\overline{X}}} \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}_{|\widetilde{\overline{S}}} \longrightarrow 0,$$

and the fact that $\mathcal{O}_{\widetilde{\overline{S}}}(\mathcal{H}) \cong \mathcal{O}_{\overline{S}}(2E + 2E_1 + E_2)$ is very ample (which one can easily verify using the fact that $2E + 2E_1 + 2E_2$ is very ample and [CD, Cor.2, page 283]) and $h^1(\mathcal{O}_{\widetilde{\overline{X}}}) = h^1(\mathcal{O}_{\overline{X}}) = 0$, we see that $|\mathcal{H}|$ is base-point free and thus defines a morphism $\varphi_{\mathcal{H}}$ such that $X = \varphi_{\mathcal{H}}(\widetilde{\overline{X}}) \subseteq \mathbb{P}^9$. Note that $\mathcal{H}^3 = (2E + 2E_1 + E_2)^2 = 16$, whence X is a threefold.

Let us see now that X is not a cone over its general hyperplane section $S := \psi(\widetilde{\overline{S}})$.

Consider the four planes $H_1, ..., H_4$ in $\mathbb{P}^3$ defined by the faces of the tetrahedron. As any sextic hypersurface in $\mathbb{P}^3$ that is double on the edges of the tetrahedron and goes through another point of H_i must contain H_i, we see that these four planes are contracted to four singular points $Q_1, \ldots, Q_4 \in \overline{X}$. Moreover their linear span $< Q_1, \ldots, Q_4 >$ in $\mathbb{P}^{13}$ has dimension 3, since the hyperplanes containing $Q_1, \ldots, Q_4$ correspond to sextics in $\mathbb{P}^3$ containing $H_1, \ldots, H_4$. Now suppose that X is a cone with vertex V. Then $Q_1, \ldots, Q_4$ project to V, whence $\dim < M, Q_1, \ldots, Q_4 > \leq 4$ and $\dim M \cap < Q_1, \ldots, Q_4 > \geq 2$. On the other hand we know that $M = < E_2 > \subset \overline{H}$, where $\overline{H}$ is a general hyperplane. Therefore we have that $Q_i \notin \overline{H}, 1 \leq i \leq 4$, whence $\dim \overline{H} \cap < Q_1, \ldots, Q_4 > = \dim M \cap < Q_1, \ldots, Q_4 > = 2$, so that $\overline{H} \cap < Q_1, \ldots, Q_4 > = M \cap < Q_1, \ldots, Q_4 >$. Now choose the projection from $M' = < E_1 > \subset \overline{H}$. If also $\pi_{M'}(\overline{X})$ is a cone then, by the same argument above, we get $\overline{H} \cap < Q_1, \ldots, Q_4 > = M' \cap < Q_1, \ldots, Q_4 >$ and therefore $\dim M \cap M' \geq 2$. But this is absurd since $\dim M \cap M' = 6 - \dim < E_1 \cup E_2 > = 6 - h^0(\mathcal{O}_{\overline{S}}(2E + E_1 + E_2)) = 0$. Therefore, X is an Enriques-Fano threefold satisfying (c).

Now let X' be the only threefold in $\mathbb{P}^9$ appearing in Bayle-Sano's list, namely an embedding, by a line bundle L', of a quotient by an involution of a smooth complete intersection Z of two quadrics in $\mathbb{P}^5$. Let S' be a general hyperplane section of X'. We claim that the hyperplane bundle $L'_{|S'}$ is 2-divisible in $\operatorname{Num} S'$. As $2E + 2E_1 + E_2$ is not 2-divisible in $\operatorname{Num} S$, this shows in particular that X does not belong to the list of Bayle-Sano.

By [Ba, §3, page 11], if we let $\pi : Z \to X'$ be the quotient map, we have that $-K_Z = \pi^*(L')$ and the K3 cover $\pi_{|S''} : S'' \to S'$ is an anticanonical surface in Z, that is a smooth complete intersection S'' of three quadrics in $\mathbb{P}^5$. Therefore, if H_Z is the line bundle giving the embedding of Z in $\mathbb{P}^5$, we have $-K_Z = 2H_Z$, whence, setting $p = \pi_{|S''}$, $H_{S''} = (H_Z)_{|S''}$, we deduce that $p^*(L'_{|S'}) \cong (\pi^* L')_{|S''} = 2H_{S''}$. Suppose now that $L'_{|S'}$ is not 2-divisible in $\operatorname{Num} S'$. Then $(L'_{|S'})^2 = 16$ and by [KL2, Prop.1] we have that $\phi(L'_{|S'}) = 3$ and it is easily seen that there are three isotropic effective divisors E, E_1, E_2 such that either (i) $L'_{|S'} \sim 2E + 2E_1 + E_2$ with $E.E_1 = E.E_2 = E_1.E_2 = 1$ or (ii) $L'_{|S'} \sim 2E + E_1 + E_2$ with $E.E_1 = 1$, $E.E_2 = E_1.E_2 = 2$. In case (i) we get that $p^*(E_2) \sim 2D$,

for some $D \in \operatorname{Pic} S''$. Since $(p^*(E_2))^2 = 0$, we have $D^2 = 0$ and, as we are on a K3 surface, either D or $-D$ is effective. Also $4H_{S''}.D = p^*(L'_{|S'}).p^*(E_2) = 8$, therefore $H_{S''}.D = 2$ and D is a conic of arithmetic genus 1, a contradiction. In case (ii) we get that $p^*(E_1 + E_2) \sim 2D'$, for some $D' \in \operatorname{Pic} S''$ with $(D')^2 = 2$ and $H_{S''}.D' = 5$. But now $|D'|$ cuts out a g^2_5 on the general element $C \in |H_{S''}|$ and this is a contradiction since C is a smooth complete intersection of three quadrics in $\mathbb{P}^4$. Therefore $L'_{|S'}$ is 2-divisible in $\operatorname{Num} S'$.

Now assume that X has a $\mathbb{Q}$-smoothing, that is ([Mi], [R1]) a small deformation $\mathcal{X} \longrightarrow \Delta$ over the 1-parameter unit disk, such that, if we denote a fiber by X_t, we have that $X_0 = X$ and X_t has only cyclic quotient terminal singularities. Let $L = \mathcal{O}_X(1)$. We have that $H^1(N_{S/X_0}) = H^1(\mathcal{O}_S(1)) = 0$, whence the Enriques surface S deforms with any deformation of X_0. Therefore we can assume, after restricting Δ if necessary, that there is an $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ such that $h^0(\mathcal{L}) > 0$ and $\mathcal{L}_{|X} = L$ (this also follows from the proof of [H, Thm.5], since $H^1(T_{\mathbb{P}^9}{}_{|X}) = 0$). Taking a general element of $|\mathcal{L}|$ we therefore obtain a family $\mathcal{S} \longrightarrow \Delta$ of surfaces whose fibers S_t belong to $|L_t|$, where $L_t := \mathcal{L}_{|X_t}$ and $S_0 = S \in |L|$ is general, whence a smooth Enriques surface with hyperplane bundle $H_0 := L_{|S_0} \sim 2E + 2E_1 + E_2$ of type (i) above. Therefore, after restricting Δ if necessary, we can also assume that the general fiber S_t is a smooth Enriques surface ample in X_t, so that (X_t, S_t) belongs to the list of Bayle [Ba, Thm.B] and is therefore a threefold like $X' \subset \mathbb{P}^9$.

Let $H_t = (L_t)_{|S_t}$. As we saw above, we have $H_t \equiv 2A_t$, for some $A_t \in \operatorname{Pic} S_t$. This must then also hold at the limit, so that $H_0 \sim 2E + 2E_1 + E_2 \equiv 2A_0$, for some $A_0 \in \operatorname{Pic} S_0$. But then E_2 would be 2-divisible in $\operatorname{Num} S_0$, a contradiction.

We have therefore shown that X does not have a $\mathbb{Q}$-smoothing. In particular it does not lie in the closure of the component of the Hilbert scheme consisting of Enriques-Fano threefolds with only cyclic quotient terminal singularities (the fact that such threefolds do fill up a component of the Hilbert scheme is a simple consequence of the fact that one can globalize, on a family, the construction of the canonical cover [Mi, Proof of Thm.4.2], [KM, 5.3]). Hence (a) is proved.

To see (b) note that $\widetilde{\overline{X}}$ is terminal (because $\overline{X}$ is), whence the morphism $\varphi_{\mathcal{H}}$ factorizes through $\widetilde{X}$. Since $\widetilde{X}$ is $\mathbb{Q}$-Gorenstein by [Ch], an easy calculation, using a common resolution of singularities of $\widetilde{\overline{X}}$ and $\widetilde{X}$ and the facts that $-K_{\widetilde{X}} \equiv \mu^*\mathcal{O}_X(1)$ and $-K_{\overline{X}} \equiv \mathcal{O}_{\overline{X}}(1)$, shows that $\widetilde{X}$ is canonical.

Finally, the same proof as above shows that $(\widetilde{X}, \mu^*\mathcal{O}_X(1))$ does not belong to the list of Fano-Conte-Murre-Bayle-Sano and that $\widetilde{X}$ does not have a $\mathbb{Q}$-smoothing. Hence by [Mi, MainThm.2] $\widetilde{X}$ cannot be terminal. This proves (b). □

Remark 16.2. Since $\widetilde{X}$ has canonical but not terminal singularities, the morphism $\varphi_{\mathcal{H}}$ in the proof of Proposition 16.1 must in fact contract divisors, for otherwise $\widetilde{X}$ would be terminal (as $\widetilde{\overline{X}}$ is). This contraction makes $\widetilde{X}$ acquire new singularities. It would be interesting to understand how these singularities affect the non existence of a $\mathbb{Q}$-smoothing. Moreover we observe that $\widetilde{X}$ is a $\mathbb{Q}$-Fano threefold of Fano index 1 with canonical singularities not having a $\mathbb{Q}$-smoothing, thus showing that Minagawa's theorem [Mi, MainThm.2] cannot be extended to the canonical case.

Remark 16.3. Somehow Proposition 16.1(c) shows the spirit of the method of classification we introduce in this paper. The question of existence of threefolds is reduced to the geometry of decompositions of the hyperplane bundle of the surface sections. In fact, in the case of Enriques surfaces, we can write down all "decomposition types" of hyperplane bundles of genus $g \leq 17$. In each case one can try to either show nonextendability or to find a threefold with that particular hyperplane section, whence either get a new one or one belonging to the list of Bayle-Sano. For instance, Prokhorov's new Enriques-Fano threefold of genus 17 must belong to one of the three cases (a1)-(a3) of Proposition 15.1. Once one proves existence one can use the same construction method as in the proof of Proposition 16.1 and project down to find new Enriques-Fano threefolds. We

also observe that our method shows that, in several "decomposition types" of hyperplane bundles of genus $g \leq 17$, we can prove that the Enriques-Fano threefold is not itself hyperplane section of some fourfold and that its general Enriques surface section must contain rational curves.

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